Algebraic Geometry

Lecture Notes

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September 28, 2025

Contents

1 Lecture I

We begin by recalling the theory of affine algebraic varieties, which we glimpsed during last year's lectures and exercises. This is useful both to build intuition and motivation for the language of schemes, and also to remember some fundamental results from commutative algebra. I advise you to look into the numerous references (the most important are listed on the Moodle page). The aim of this first lecture is also to define some of the main objects which we shall later study via the language of schemes.

1.1 Affine algebraic varieties

Let K be an algebraically closed field, which we fix from now on. The weak Nullstellensatz says that we can identify K^n with the set of maximal ideals of $K[x_1, \dots, x_n]$, and it clearly expresses the first fundamental bridge between algebra and geometry. The full Nullstellensatz instead says that for every ideal $I \subset K[x_1, \dots, x_n]$ we have

$$I(V(I)) = \sqrt{I}.$$

(Why does this imply the weak form?)

Recall that we can endow K^n with the Zariski topology, where the closed subsets are of the form $V(I)$ for $I \subset K[x_1, \dots, x_n]$. We also call the subsets $V(I)$ the *algebraic subsets* of K^n. These are very explicitly defined: since $K[x_1, \cdots, x_n]$ is Noetherian, every ideal is finitely generated. Write $I = (f_1, \dots, f_d)$, then

$$V(I) = \{(x_i) \in K^n : \ f_1(x_1, \dots, x_n) = \cdots = f_d(x_1, \dots, x_n) = 0\},$$

which in turn corresponds to maximal ideals $\mathfrak{m} \subset K[x_1, \dots, x_n]$ with $I \subset \mathfrak{m}$.

These subsets are the basic building blocks of algebraic geometry, and we shall spend some time understanding their properties. First of all, we can endow $V(I)$ with a topology in two equivalent ways (verify this): either restrict the Zariski topology of K^n to $V(I)$, or identify $V(I)$ with the set of maximal ideals $\mathfrak{m} \subset A := K[x_1, \dots, x_n]/I$ and endow it with the topology generated by

$$V(J) := \{\mathfrak{m} \subset A : \ J \subset \mathfrak{m}\}$$

for ideals $J \subset A$.

The topology of $V(I)$ is very different from what we are normally used to, and enjoys many finiteness properties:

Definition 1.1. A topological space X is said to be *Noetherian* if any descending chain of closed subsets

$$Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n \supseteq \cdots$$

eventually stabilizes.

The fact that $K[x_1, \cdots, x_n]$ is a Noetherian ring then readily implies that any $V(I)$ is also Noetherian. Recall that we defined the topological space $\mathrm{Spec}(A)$ for any commutative ring. Show that if A is Noetherian then $\mathrm{Spec}(A)$ is Noetherian (is this an if and only if?). Note, on the other hand, that the usual topology of $\mathbb{R}^n$ is never Noetherian for $n \geq 1$.

Theorem 1.2 (Noetherian induction). *Let X be a Noetherian topological space and P be a property of the closed subsets of X. Assume $P(\emptyset)$ is true, and that for every closed subset $Z \subset X$, the fact that $P(W)$ is true for every proper closed $W \subsetneq Z$ implies that also $P(Z)$ is true. Then P is true for all closed subsets of X.*

This or similar formulations appear many times in proofs, for example when studying coherent cohomology. We use it now to show that every $V(I)$ can be decomposed uniquely into a union of irreducible subsets:

Definition 1.3. A Noetherian topological space X is *irreducible* if for every two closed subsets $X_1, X_2 \subsetneq X$ we have $X_1 \cup X_2 \subsetneq X$.

Recall that $V(I)$ is irreducible if and only if $\sqrt{I}$ is prime.

Theorem 1.4. *Every closed subset $V(I) \subset K^n$ can be decomposed uniquely as a finite union of distinct irreducible closed subsets.*

Proof. Prove it using Noetherian induction. Compare it to the proof that every radical ideal of $K[x_1, \dots, x_n]$ decomposes uniquely into a product of prime ideals. We omit the details. □

The subsets $V(I)$ with I prime are called *affine algebraic varieties.*

Remark 1.5. To gain geometric intuition for algebraic varieties, assume that $\mathrm{char}(K) = 0$ and that $|K| \leq |\mathbb{C}|$ (this assumption is very reasonable). Then one can embed $K \subset \mathbb{C}$ and consider $V(I)$ as an algebraic subset of $\mathbb{C}^n$. But this is also closed for the usual topology, and we can study its properties with tools from topology, real analysis, and complex analysis. As one will see, many topological properties of $V(I)$ can be obtained by purely algebraic methods without leaving the language of algebraic geometry.

Many times we can also embed $K \subset \mathbb{R}$, and $V(I)$ yields a closed subset of $\mathbb{R}^n$. Although this can still be used to get geometric insights into $V(I)$, we remark that $V(I)$ may behave strangely (e.g. it may be empty). This is why classical algebraic geometry is formulated over algebraically closed fields: so that we see all points of $V(I)$ over K.

The topology of irreducible Noetherian topological spaces is very coarse:

Proposition 1.6. *If X is an irreducible topological space and $U \subset X$ is a non-empty open subset, then U is dense.*

Proof. Assume that $\overline{U} \neq X$, so $V = X \setminus \overline{U}$ is open and non-empty. Note that $V \cap U = \emptyset$ and that $X = \overline{U} \cup \overline{V}$, contradicting irreducibility. □

Thus affine algebraic varieties are never Hausdorff (T_2) in the classical sense. In fact, the right concept is the one of *separatedness*, as we shall see later in the course.

Observation 1.7. *If we wish to have a faithful dictionary between algebra and geometry, we already see that something is amiss: different ideals can yield the same algebraic subset. Thus the algebraic side carries more information than the geometric side in classical algebraic geometry. This might seem like a minor inconvenience at first, but the deeper one digs into the structure of algebraic varieties, the more essential it becomes. The interpretation of this phenomenon in terms of nilpotents in the structure sheaf of algebraic varieties—and in general the realization of the key role played by nilpotent elements in algebraic geometry—is one of the leading motivations for the abstract language of schemes.*

1.1.1 Examples

At this stage, we cannot say much more, but let us list some basic examples:

1. **Linear varieties.** These are the zero sets of degree one polynomials. They can always be written as $P + W$, where $P \in K^n$ is a point and $W \subset K^n$ is a vector subspace.

2. **Conics in $\mathbb{A}^2_K$.** Assume that $\mathrm{char}(K) \neq 2$ and let $Q(x, y) \in K[x, y]$ be a quadratic polynomial. Then $C = V(Q) \subset K^2$ is called an affine conic. When is C irreducible? If C is not irreducible, in which ways can it decompose?

3. **The Fermat curve.** For $n \geq 1$ let

$$F_n = V(x^n + y^n - 1) \subset K^2.$$

   Show that F_n is always irreducible (what happens when $\mathrm{char}(K)$ divides n?). Fermat's Last Theorem then says that the only $(a, b) \in \mathbb{Q}^2$ satisfying $a^n + b^n = 1$ are $(1, 0)$ and $(0, 1)$. In fact, much of algebraic number theory boils down to the study of polynomial equations over non-closed fields. Since modern algebraic geometry works with any scheme as a base (not necessarily a field), its language automatically incorporates many classical concepts of algebraic number theory (e.g. ideal class groups, ramification theory, etc.). It will eventually allow one to show how the topological properties of a curve $C \subset \mathbb{C}^2$ (or, more generally, of an algebraic variety) influence the set $C \cap \mathbb{Q}^2$, assuming that C can be defined by equations with coefficients in $\mathbb{Q}$.

4. **Hypersurfaces.** A hypersurface is by definition $V(f) \subset K^n$ where $f \in K[x_1, \dots, x_n]$ is an irreducible polynomial. These are heavily studied for many reasons: they are easy to define, yet they already give very rich families of examples (in fact, as we shall see, every variety is birational to some hypersurface).

1.1.2 Localization and open subsets

Let us now recall the operation of localization and how to describe the Zariski open subsets of a variety $X \subset K^n$. Let R be any commutative ring and let $S \subset R$ be a multiplicative subset. We defined the localization $S^{-1}R$ last year: this is a ring together with a natural map

$$\phi : R \to S^{-1}R$$

which is universal among ring maps $f : R \to R'$ such that $f(S) \subset (R')^\times$, i.e. every $f(s)$ is invertible for $s \in S$. We have shown that the induced map

$$\mathrm{Spec}(S^{-1}R) \longrightarrow \mathrm{Spec}(R)$$

is an open immersion with image

$$\{\mathfrak{p} \in \mathrm{Spec}(R) : S \cap \mathfrak{p} = \emptyset\}.$$

The two most important multiplicative sets in algebraic geometry are:

- $S = \{1, f, f^2, \dots\}$ for some $f \in R$, in which case we write $S^{-1}R = R_{(f)}$;
- $S = R \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, in which case we write $S^{-1}R = R_{\mathfrak{p}}$.

Recall that if R is a domain, then $S^{-1}R \subset \mathrm{Frac}(R)$ via the universal property of localization. Concerning $R_{(f)}$, we have a natural identification

$$R_{(f)} \cong \{\mathrm{Frac} h f^n : h \in R, \ n \geq 0\} \subset \mathrm{Frac}(R).$$

Now let $A = K[x_1, \dots, x_n]/\mathfrak{p}$ and let $X = V(\mathfrak{p}) \subset K^n$ be the associated algebraic variety. For any $f \in A$ consider the open subset $D_f = X \setminus V(f)$. These are called *principal open subsets*.

Lemma 1.8. *The following hold:*

1. *The open sets D_f generate the Zariski topology of X;*
2. *Every D_f has a natural structure of an affine algebraic variety.*

Proof. Point (1) is an exercise. For (2), write $\mathfrak{p} = (f_1, \dots, f_k) \subset K[x_1, \dots, x_n]$, so that

$$R_{(f)} \cong K[x_1, \dots, x_n, y]/(f_1, \dots, f_k, yf - 1).$$

Thus $R_{(f)}$ is itself a finitely generated K-algebra, and hence defines an affine algebraic variety in K^{n+1}. Its points correspond uniquely to maximal ideals of $R_{(f)}$, and hence to points of D_f. □

In general, if $X \subset K^n$ is an algebraic variety and $U \subset X$ is an open subset, we say that $U \subset K^n$ is a quasi affine variety.

Excercise 1.9. Show that $\mathrm{GL}_n(K) \subset M_n(K)$ has a natural structure of an affine algebraic variety, where we identify $M_n(K) \cong K^{n^2}$.

1.2 Dimension theory

Recall the definition of the Krull dimension of a ring R: it is the maximal length of a chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

(if such a maximum exists). Similarly, for a topological space X, we define $\dim(X)$ as the maximal length of a chain of non-empty irreducible closed subsets

$$Z_n \subsetneq \cdots \subsetneq Z_1 \subsetneq Z_0.$$

So if $A = K[x_1, \cdots, x_n]/I$ is a finitely generated K-algebra $\dim(A) = \dim(V(I))$. We also proved that if A is an integral finitely generated K-algebra then

$$\dim(A) = \mathrm{tr.deg.}_K(\mathrm{Frac}(A)).$$

One would like to have a well-defined notion of codimension. Clearly we could just define $\mathrm{codim}(Z) = \dim(X) - \dim(Z)$ where $Z \subset X$ is a closed subset, but this is an ad-hoc definition which is not well-behaved in practice. The only thing we can do algebraically is the following:

Definition 1.10. Let $\mathfrak{p} \subset R$ be a prime. Then its height $\mathrm{ht}(\mathfrak{p})$ is the maximal length of a chain of prime ideals $\mathfrak{p}_n \subsetneq \cdots \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}$ (if it exists).

The problem now is to show that

$$\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R).$$

But this is not true for general rings, and the right notion to consider is the following:

Definition 1.11. A ring R is *catenary* if for any two primes $\mathfrak{p}_1 \subset \mathfrak{p}_2$, all maximal chains of prime ideals starting at $\mathfrak{p}_1$ and ending at $\mathfrak{p}_2$ have the same finite length.

It is easy to see that any localization or quotient of a catenary ring is still catenary (prove it). The first counterexample of a Noetherian ring which is not catenary was constructed by Nagata (he also constructed the first example of a Noetherian ring with infinite Krull dimension). On the other hand, most if not all of the rings appearing in algebraic geometry are catenary:

Theorem 1.12. *If A is a finitely generated integral K-algebra then it is catenary.*

Proof. The proof can be found in Matsumura's book. We advise the reader to take a look at it. □

Note that without the integrality condition the result clearly fails. For example, consider

$$A = K[x, y, z]/(xy, xz).$$

Then A is not integral (picture it in K^3). One can check that both chains

$$(y, z) \subset (x, y, z), \qquad (x) \subset (x, y) \subset (x, y, z)$$

are maximal chains of prime ideals of different lengths. What is happening geometrically?

Proposition 1.13. *Let R be integral, catenary, and of finite dimension. Then for any prime $\mathfrak{p} \subset R$ we have*

$$\dim(R) = \mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}).$$

Proof. The proof follows by concatenating chains of prime ideals starting with (0) and ending with $\mathfrak{p}$ with chains of prime ideals in $R/\mathfrak{p}$, and using catenarity to ensure equality of lengths. □

We shall return to the notion of dimension again later in the course. For the time being, consider the following:

Example 1.14. Let $(R, \mathfrak{m})$ be a discrete valuation ring with uniformizer $\pi \in \mathfrak{m}$ fraction field K and residue field k. Consider the polynomial ring $R[x]$. Then, there are two kind of maximal ideals $m \subset R[x]$:

1. Either $m \cap R = \mathfrak{m}$ or
2. $m \cap R = (0)$.

For example, the ideal $(\pi, x-r)$ for $r \in R$ belongs to case (1) whereas $(\pi x - 1) \subset R[x]$ to case (2). In particular $R[x]$ is not catenary.

1.3 Morphisms of varieties

Every time we define new objects in mathematics we also have to define morphisms between them. We have two ways to do so in the case of affine algebraic varieties: let $X \subset K^n$ and $Y \subset K^m$ be two varieties:

1. A map $f : X \to Y$ is a morphism if there are $P_1, \cdots, P_m \in K[x_1, \cdots, x_n]$ such that $f(x) = (P_1(x), \cdots, P_m(x))$ for every $x \in X$.
2. A map $f : X \to Y$ is a morphism if for every $P \in X$ there are polynomial functions $H, G : K^n \to K^m$ such that $G(u) \neq 0$ and $f(u) = H(u)/G(u)$ for every $u \in U \cap X$.

The two definitions turn out to be equivalent, but a priori the first definition is only contained in the second, and it is more rigid as it forces all morphisms to be restrictions of 'global' polynomial maps between K^n and K^m. For example, the first definition does not work for quasi-affine subsets, but the second does. Thinking more about the difference between the two definitions naturally leads to the notion of sheaves. We adopt the first definition for the time being. Finally, we say that X and Y are isomorphic if there are morphisms $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$.

- Show that any $f \in \mathrm{Hom}(X, Y)$ is continuous for the Zariski topology.
- Since K is a ring $\mathrm{Mor}(X, K)$ is also a ring. Show that $\mathrm{Mor}(X, K) \cong A$ where $X = V(I) \subset K^n$ and $A = K[x_1, \cdots, x_n]/\sqrt{I}$.
- In particular, take $f \in A$. Then $\mathrm{Mor}(D_f, K) \cong R_{(f)}$, where we give D_f the structure of an algebraic variety as before.
- (Linear projections) Let W be a finite dimensional K-vector space. Let $\phi : K^n \to W$ be a linear surjective morphism and let $X \subset K^n$ be an algebraic variety. Then the restriction $\phi_{|X}$ is called a linear projection from X to W.

- (Projections from a point) Choose a point $P \in K^n$ and a linear hyperplane $H \subset K^n$ (that is, the translate of a linear subspace $V \subset K^n$ of dimension $n-1$ or, equivalently, the zero set of a degree one polynomial $F \in K[x_1, \cdots, x_n]$) such that $P \notin H$. For any point $x \in K^n \setminus P$ let ℓ_x be the line joining P and x. Then only two things can happen: either $\ell_x \cap H$ consists of one point or $\ell_x \cap H = \emptyset$. This second case happens only if $x \in H'$, the unique hyperplane parallel to H with $P \in H'$. So on $D_{H'} = K^n \setminus H'$ we obtain a well-defined set-theoretic map $D_{H'} \to H$, called the projection from P to H. Show that this is a morphism of algebraic varieties. We can restrict this to any algebraic variety and obtain a morphism defined on a principal open subset.

- (Affine conics, continued). Consider an irreducible conic $C = V(Q) \subset K^2$ and assume that $(0,0) \in C$, which is always possible up to translation. We want to show that either C is isomorphic to K or to $K \setminus 0 = D_x$ as algebraic varieties. In fact, we can do this in two ways. To prove the result algebraically, write $Q = Q_2 + Q_1$ where Q_2 is homogeneous of degree 2 and Q_1 is homogeneous of degree 1 (there is no constant term since $(0,0) \in Q$). So $Q_2(x,y) = ax^2 + bxy + cx^2$ which is the quadratic form associated to the bilinear form $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ (recall that $\mathrm{char}(K) \neq 2$). Then Q_2 as a matrix has rank either one or two. With a linear change of variable, show that in the first case C is isomorphic to the parabola $V(y - x^2) \cong K$ and that in the second one to the hyperbole $V(xy-1) \cong K \setminus 0$.

  One can also prove the result geometrically, but the light on this will be shed once we introduce projective spaces, and see the conic as a curve inside the projective plane. One begins by answering the following: how many lines $L \subset K^2$ passing through $(0,0)$ are such that $L \cap C = \{(0,0)\}$? Clearly the tangent line $T \subset K^2$ is one of them. Show that there are either one or two more, and fix one of them and call it L. Finally, for any point $P' \in C$ let L' be the unique line parallel to L and passing through P'. Show that $L' \cap T$ consists of a unique point $\phi(P')$. Show that this defines a morphism $\phi : C \to T$. Show that in the first case (where there is only one line L) the map is an isomorphism, and in the second, the map is an isomorphism onto the complement of a point.

- (Frobenius) Assume that $\mathrm{char}(K) = p > 0$ and let $X = V(\mathfrak{p}) \subset K^n$ be a variety. Assume for simplicity that $\mathfrak{p} = (f_1, \cdots, f_d)$ where $f_i \in \mathbb{F}_p[x_1, \cdots, x_n] \subset K[x_1, \cdots, x_n]$ for every i. Now consider the map $F : K^n \to K^n$ sending $(a_1, \cdots, a_n) \mapsto (a_1^p, \cdots, a_n^p)$, which corresponds to the K-algebra map $K[x_1, \cdots, x_n] \to K[x_1, \cdots, x_n]$ sending $x_i \mapsto x_i^p$. Show that F induces a bijection $F : X \to X$, which is called the geometric Frobenius of X. Note that we would expect the map $F : X \to X$ to have degree $p^{\dim(X)}$ (for instance, try to show that the induced map of function fields $\mathrm{Frac}(A(X)) \to \mathrm{Frac}(A(X))$ yields a finite extension of degree $p^{\dim(X)}$) but topologically the map is a bijection, hence it should be an isomorphism. This 'problem' is also solved by schemes, for which indeed the map turns out to have the right degree.

Now, let $X \subset K^n$ be given by $V(\mathfrak{p})$ and $Y \subset K^m$ be given by $V(\mathfrak{q})$.

Proposition 1.15. *There is a natural identification*

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}_{\text{K-}alg}(A(Y), A(X))$$

where, from now on, we write $A(X) = \mathrm{Hom}(X, K)$.

Proof. Take a morphism $f : X \to Y$. Then the induced map $A(Y) \to A(X)$ is simply given by composition

$$A(Y) \ni (g : Y \to K) \mapsto (g \circ f : X \to K) \in A(X),$$

which defines a ring morphism.

For the other direction, write $A(X) = K[x_1, \cdots, x_n]/\mathfrak{p}$ and $A(Y) = K[x_1, \cdots, x_m]/\mathfrak{q}$. Consider a map of rings $\phi : A(Y) \to A(X)$; this induces a map of rings $K[x_1, \cdots, x_m] \to A(X)$ which sends x_i to some $f_1' \in A(X)$. Now $A(X)$ is quotient of $K[x_1, \cdots, x_n]$. Find $f_i \in K[x_1, \cdots, x_n]$ which reduce to $f_i' \in A(X)$; by freeness of $K[x_1, \cdots, x_n]$ we can thus define define a morphism $K[x_1, \cdots, x_m] \to K[x_1, \cdots, x_n]$ hence a morphism $f : K^n \to K^m$ sending $(k_1, \cdots, k_n) \mapsto (f_1(k_1, \cdots, k_n), \cdots, f_m(k_1, \cdots, k_n))$. The facet that $f(X) \subset Y$ then follows from the commutativity of

$$\begin{array}{ccc} K[x_1, \cdots, x_m] & \longrightarrow & K[x_1, \cdots, x_n] \\ \downarrow & & \downarrow \\ A(Y) & \longrightarrow & A(X) \end{array}$$

These constructions are one the inverse of the other. □

2 Lecture II / III

2.1 Presheaves

The philosophy behind sheaves comes precisely from the two different possible definitions of morphisms between algebraic varieties. Although the concept of sheaf is not always stated explicitly, it is already present in many constructions in analysis. For example, let $U \subset \mathbb{C}^n$ be an open subset (with the usual topology) and let $f : U \to \mathbb{C}$ be a continuous function. Then f is said to be holomorphic on U if, for every point $P \in U$, there exists an open neighborhood $V \subset U$ of P over which f can be expressed as a converging power series. In other words, a holomorphic function is defined by its *local behaviour*, and typically we cannot express f globally on all of U as a single converging power series. This illustrates the principle that global objects are determined by compatible local data, which is exactly what the concept of a sheaf formalizes.

Definition 2.1. Let X be a topological space. A *presheaf* of abelian groups $\mathscr{F}$ on X consists of the following data:

1. For every open subset $U \subset X$, an abelian group $\mathscr{F}(U)$;
2. For every inclusion of open sets $V \subset U$, a *restriction map*

$$\rho_{U,V} : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$$

   satisfying the following:

   (a) $\rho_{U,U} = \mathrm{Id}_{\mathscr{F}(U)}$;
   (b) For any chain $U_1 \subset U_2 \subset U_3$, we have $\rho_{U_2,U_1} \circ \rho_{U_3,U_2} = \rho_{U_3,U_1}$.

An element $s \in \mathscr{F}(U)$ is called a *section* of $\mathscr{F}$ over U, and we write

$$s|_V := \rho_{U,V}(s)$$

for its restriction to $V \subset U$. The group $\mathscr{F}(X)$ is often denoted $\Gamma(X, \mathscr{F})$ and its elements are called *global sections*.

Remark 2.2. Similarly, one can define presheaves of sets, rings, or other objects. In fact, for any category $\mathscr{C}$, a presheaf with values in $\mathscr{C}$ can be defined in the same way. Equivalently, let $\mathrm{Open}(X)$ be the category whose objects are open subsets of X, with

$$\mathrm{Hom}(U, V) = \begin{cases} \{*\}, & \text{if } U \subset V, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Then a presheaf with values in $\mathscr{C}$ is exactly a contravariant functor

$$\mathscr{F} : \mathrm{Open}(X) \longrightarrow \mathscr{C},$$

where $\mathscr{F}(U)$ is the value on U and the restrictions are given by the functorial action.

2.1.1 Examples

We now make some important examples of presheaves, divided into two classes.

1. Structure sheaves of geometric objects.

   All the following are presheaves of rings, with restriction maps given by restricting functions to smaller open sets.

   (a) *Continuous functions:* Let X be a topological space. Define

   $$\mathscr{C}_X(U) = \{f : U \to \mathbb{R} \mid f \text{ is continuous}\}.$$

   (b) *Differentiable functions:* Let X be a $\mathscr{C}^\infty$-manifold. Define

   $$\mathscr{C}_X^\infty(U) = \{f : U \to \mathbb{R} \mid f \text{ is smooth}\}.$$

   (c) *Holomorphic functions:* Let X be a complex manifold. Define

   $$\mathscr{O}_X^{\mathrm{hol}}(U) = \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\}.$$

   (d) *Regular functions:* Let $X \subset K^n$ be an algebraic variety with the Zariski topology. Define

   $$\mathscr{O}_X(U) = \left\{ f : U \to K \;\middle|\; \begin{array}{c} \text{for every } P{\in}U \text{ there is an open } P{\in}V{\subset}U \text{ and} \\ \text{polynomials } h,g{\in}K[x_1,\cdots,x_n] \text{ such that} \\ g(v){\neq}0 \text{ and } f(v){=}h(v)/g(v) \text{ for every } v{\in}V \end{array} \right\}.$$

2. Constant presheaves.

   (a) Let G be an abelian group. The constant presheaf with values in G is

   $$\underset{\sim}{G}(U) = \{f : U \to G \mid f \text{ is constant}\}.$$

   (b) Endow G with the discrete topology. The *locally constant presheaf* $\underline{G}$ is defined by

   $$\underline{G}(U) = \{f : U \to G \mid f \text{ is continuous}\}.$$

(c) Let $\pi : Y \to X$ be a covering map of topological spaces. Define the presheaf of sections

$$\mathcal{S}_{\mathfrak{p}}i(U) = \{f : U \to Y \mid f \text{ continuous and } \pi \circ f = \mathrm{Id}_U\}.$$

We now discuss some differences and similarities between the examples of presheaves introduced before. The first and most important observation is that all the presheaves in point (1) satisfy some extra fundamental properties.

Let $\mathcal{O}$ be any of the presheaves from point (1), let $U \subset X$ be an open subset, and let $\{U_i\}_{i\in I}$ be any open cover of U. Then the following hold:

I. **(Gluing of functions)** Given sections $f_i \in \mathcal{O}(U_i)$ such that

$$f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j} \quad \text{for all } i, j \in I,$$

there exists a section $f \in \mathcal{O}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

II. **(Local nature of functions)** If $f, g \in \mathcal{O}(U)$ are such that

$$f|_{U_i} = g|_{U_i} \quad \text{for all } i \in I,$$

then $f = g$.

These two properties are exactly what distinguish *sheaves* from general presheaves. Let us see how these properties can fail:

Failure of gluing (I). Consider a topological space $X = X_1 \sqcup X_2$ with $X_1, X_2 \neq \emptyset$, and let $\tilde{G}$ be the constant presheaf defined earlier. Then

$$\tilde{G}(X_1) = G, \quad \tilde{G}(X_2) = G, \quad \tilde{G}(\emptyset) = 0.$$

Any two sections $g_1 \in \tilde{G}(X_1)$ and $g_2 \in \tilde{G}(X_2)$ satisfy the local compatibility condition, but if $g_1 \neq g_2$, they cannot be glued to a global section in $\Gamma(X, \tilde{G})$. In contrast, the locally constant presheaf $\underline{G}$ satisfies the gluing property and we have $\underline{G}(X) = G^{\pi_0(X)}$ (at lease when all connected components of X are open).

Failure of locality (II). To construct a presheaf that fails property (II) is slightly more subtle. Consider the presheaf of bounded continuous functions

$$\mathcal{C}_X^b(U) = \{f : U \to \mathbb{R} \mid f \text{ is bounded}\} \subset \mathcal{C}_X(U).$$

Then the quotient presheaf

$$\mathcal{F}(U) = \mathcal{C}_X(U)/\mathcal{C}_X^b(U)$$

does not satisfy (II). For example, if $X = \mathbb{R}$, consider the section represented by $x \in \mathcal{C}_X(\mathbb{R})$. Then $0 \neq x \in \mathcal{F}(\mathbb{R})$, but for any open cover $\{U_i\}_{i\in I}$ of $\mathbb{R}$ such that each $\overline{U_i}$ is compact we have $x|_{U_i} = 0$ in $\mathcal{F}(U_i)$.

2.2 Sheaves

Definition 2.3. A presheaf $\mathscr{F}$ is called a *sheaf* if for every open subset $U \subset X$ and every open cover $\{U_i\}_{i \in I}$ of U, the following hold:

I. **(Gluing of sections)** Given sections $s_i \in \mathscr{F}(U_i)$ such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \text{for all } i, j \in I,$$

there exists a section $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

II. **(Locality)** If $s, s' \in \mathscr{F}(U)$ satisfy

$$s|_{U_i} = s'|_{U_i} \quad \text{for all } i \in I,$$

then $s = s'$.

Remark 2.4. Property (II) guarantees that the gluing in (I) is unique.

Now, there is a canonical way to obtain a sheaf from a presheaf but first we need talk about stalks (note: the terminology sheaves and stalks is not random). Stalks are used to study sections around a specific point $P \in X$:

Definition 2.5. Let $\mathscr{F}$ be a presheaf and let $P \in X$. The *stalk* of $\mathscr{F}$ at P is

$$\mathscr{F}_P = \varinjlim_{P \in U} \mathscr{F}(U),$$

where the colimit runs over all open neighborhoods U of P. An element of $\mathscr{F}_P$ is called a *germ* of a section at P. Similarly, for a subset $Z \subset X$, the stalk at Z is defined by

$$\mathscr{F}_Z = \varinjlim_{Z \subset U} \mathscr{F}(U).$$

Remark 2.6. In practice, a germ in $\mathscr{F}_P$ is represented by a pair (U, s) with $P \in U$ and $s \in \mathscr{F}(U)$. Two such pairs (U, s) and (U', s') define the same germ if there exists an open $V \subset U \cap U'$ containing P such that $s|_V = s'|_V$. If $\mathscr{F}$ takes values in a category $\mathscr{C}$, we assume that $\mathscr{C}$ admits direct limits so that this definition makes sense.

Remark 2.7. For the sheaves of functions introduced earlier, the restriction maps $\rho_{U,V}$ have special properties: they are surjective for $\mathscr{C}_X$ and $\mathscr{C}_X^\infty$ (due to the existence of partitions of unity), and injective for $\mathscr{O}_X^{\text{hol}}$ and $\mathscr{O}_X$ (because holomorphic and regular functions are rigid: if they agree on an open subset, they agree everywhere).

Let now $X = V(\mathfrak{p}) \subset K^n$ be a variety, and consider the sheaf $\mathcal{O}_X$ more closely. Let $K(X) = \mathrm{Frac}(A(X))$.

Lemma 2.8. *For any open subset $U \subset X$, there is a natural inclusion*

$$\mathcal{O}_X(U) \subset K(X).$$

Proof. Let $f \in \mathcal{O}_X(U)$. By definition, there exists a finite open cover U_i of U and polynomials $P_i, Q_i \in K[x_1, \dots, x_n]$ such that $Q_i(u) \neq 0$ for all $u \in U_i$ and $f(u) = P_i(u)/Q_i(u)$ on U_i. Now $Q_i \notin \mathfrak{p}$ for otherwise $Q_i(x) = 0$ for every $x \in X$. Hence Q_i is invertible in the localization $K[x_1, \cdots, x_n]_{\mathfrak{p}}$ and so it defines an element $[Q_i] \in K(X) = K[x_1, \cdots, x_n]_{\mathfrak{p}}/\mathfrak{p}$. We claim that $[P_i][Q_i]^{-1} \in K(X)$ is well-defined and independent of i. First, if $P_i(u)/Q_i(u) = P_i'(u)/Q_i'(u)$ for every $u \in U_i$ then $P_iQ_i' - P_i'Q_i$ defines an element of $A(X)$ whose vanishing locus contains U_i, which is a non-trivial Zariski open. If $P_iQ_i' - P_i'Q_i$ was non-zero in $A(X)$ then also its non-vanishing locus would be a non-empty Zariski open. Since any two Zariski opens intersect because X is irreducible, we see that $P_iQ_i' - P_i'Q_i \in \mathfrak{p}$ and thus $[P_i][Q_i]^{-1} \in K(X)$ does not depend on the representatives P_i, Q_i over U_i. Similarly (i.e., using the fact that $U_i \cap U_j \neq \emptyset$) one shows that $[P_i][Q_i]^{-1} = [P_j][Q_j]^{-1}$ as well.

This defines a map $\mathcal{O}_X(U) \to K(X)$. Suppose that this were not injective. Then $[P_i] \in A(X)$ must vanish on a non-trivial Zariski open - hence it vanishes everywhere - and therefore $f = 0$. □

Remark 2.9. This also shows that the restriction maps $\rho_{U,V}$ for $\mathcal{O}_X$ are injective whenever $V \neq \emptyset$.

Theorem 2.10. *Let $X = V(\mathfrak{p}) \subset K^n$ be as before.*

1. *For every $x \in X$ the stalk $\mathcal{O}_{X,x}$ is isomorphic to the localization $A(X)_{\mathfrak{m}_x}$.*
2. *We can identify*
$$\mathcal{O}_X(U) = \bigcap_{x \in U} A(X)_{\mathfrak{m}_x} \subset K(X)$$
3. *In particular, $\Gamma(X, \mathcal{O}_X) = A(X)$ (as promised).*

Proof.

- Using the previous lemma we have
$$\mathcal{O}_{X,x} = \{f/g : f, g \in A(X) \text{ and } g(x) \neq 0\} \subset K(X).$$
- The inclusion $\mathcal{O}_X(U) \subset \bigcap_{x \in U} A(X)_{\mathfrak{m}_x}$ is clear. Let now h be in the intersection, then for every $x \in U$ we can write $f = h_x/g_x$ for some

$f_x, g_x \in A(X)$ such that $g_x(x) \neq 0$. So g_x does not vanish in a neighbrhood $x \in U_x \subset X$ of x. But then there are finitely many $x_1, \cdots, x_n$ such that $U_i = U_{x_i}$ cover X and such that $g_i(u) \neq 0$ for every $u \in U_i$. Thus, the function $U \to K$ sending $u \mapsto f_i(u)/g_i(u)$ if $u \in U_i$ is well-defined and defines a section of $\mathcal{O}_X(U)$.

- Let $h \in \bigcap_{x \in X} A(X)_{\mathfrak{m}_x}$ and consider the ideal $I = \{g \in A(X) : gh \in A(X)\} \subset A(X)$. If this is not trivial, then it is contained in a maximal ideal $\mathfrak{m} \subset A(X)$ corresponding to some $x \in X$. But $h \in A(X)_{\mathfrak{m}}$ so $h = f/g$ with $f \in A(X)$ and $g(x) \neq 0$ and hence also $g \in I$, contradicting that $I \subset \mathfrak{m}$.

□

The fact that a sheaf is determined by its local behavious is captured in a number of propositions, for example:

Proposition 2.11. *If $\mathcal{F}$ is a sheaf then for every open $U \subset X$ the natural map*

$$\mathcal{F}(U) \to \prod_{P \in U} \mathcal{F}_P$$

is injective.

Proof. If a $s \in \mathcal{F}(U)$ is in the kernel of the above, then by the definition of stalks, we can find for every point P an open $P \in U_P$ such that $s_{|U_P} = 0$. Since $\{U_P\}_P$ covers X the result follows by condition (II). □

Note that in fact one can also characterize the image:

Proposition 2.12. *If $\mathcal{F}$ is a sheaf then the image of*

$$\mathcal{F}(U) \to \prod_{P \in U} \mathcal{F}_P$$

corresponds to

$$\left\{ (s_P) \in \prod_{P \in U} \mathcal{F}_P : \begin{array}{c} \textit{for every } P \textit{ there is } V \subset U \textit{ open with} \\ P \in V \textit{ and a section } S \in \mathcal{F}(V) \textit{ such} \\ \textit{that } s_Q = S_Q \textit{ in } \mathcal{F}_Q \textit{ for every } Q \in V \end{array} \right\}.$$

Proof. Use the previous proposition and the glueing condition. □

Sheafification So now it is easy to associate a sheaf to any presheaf

Definition 2.13. If $\mathscr{F}$ is a presheaf its sheafification $\mathscr{F}^+$ is defined by

$$\mathscr{F}^+(U) := \left\{ (s_P) \in \prod_{P \in U} \mathscr{F}_P : \begin{array}{c} \text{for every } P \text{ there is } U \subset X \text{ open with} \\ P \in U \text{ and a section } S \in \mathscr{F}(U) \text{ such} \\ \text{that } s_Q = S_Q \text{ in } \mathscr{F}_Q \text{ for every } Q \in U \end{array} \right\}.$$

This is a sheaf and there is a natural map of presheaves $\mathscr{F} \to \mathscr{F}^+$ which is an *isomorphism* on stalks. It satisfies a universal property: for any sheaf $\mathscr{G}$ any morphism $\mathscr{F} \to \mathscr{G}$ factorises uniquely as $\mathscr{F} \to \mathscr{F}^+ \to \mathscr{G}$.

Remark 2.14. To get another glimpse into the local nature of sheaves, consider the following example: let $\mathscr{P}$ be a property on the open subsets of X such that if $U \subset V$ and $\mathscr{P}(V)$ is true then also $\mathscr{P}(U)$ is true. Define the presheaf of sets $\mathscr{F}(U) = \{*\}$ if $\mathscr{P}(U)$ is true and $\mathscr{F}(U) = \emptyset$ if $\mathscr{P}(U)$ is false and let $\mathscr{F}^+$ be its sheafification. Then $\Gamma(X, \mathscr{F}) = \{*\}$ if and only if $\mathscr{P}(X)$ is true whereas $\Gamma(X, \mathscr{F}^+) = \{*\}$ if and only if $\mathscr{P}$ is *locally true* over X, that is, X can be covered by open subsets U_i such that $\mathscr{P}(U_i)$ is true for every i.

2.3 Exact properties of sheaves and presheaves

Definition 2.15. Let $\mathscr{F}, \mathscr{G}$ be presheaves on X of sets, abelian groups, etc. A morphism $\phi : \mathscr{F} \to \mathscr{G}$ is by definition a collection of maps of sets, abelian groups, etc $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ for every open $U \subset X$ satisfying the obvious compatibility with the restriction maps.

1. ϕ is injective if ϕ_U is injective for every U;
2. ϕ is surjective if ϕ_U is surjective for every U.

Assume now that both $\mathscr{F}$ and $\mathscr{G}$ are presheaves of abelian groups:

1. $\ker(\phi)$ is the presheaf $U \mapsto \ker(\phi_U)$;
2. $\operatorname{coker}(\phi)$ is the presheaf $U \mapsto \operatorname{coker}(\phi_U)$
3. $\operatorname{Im}(\phi)$ is the presheaf $U \mapsto \operatorname{Im}(\phi_U)$.

Remark 2.16. One can form a category of presheaves of abelian groups on X which turns out to be an abelian category. The definitions above then corresponds to the abstract definitions of the same objects in any abelian category.

Remark 2.17. Recall that direct limits preserve exactness. In particular, if $\phi : \mathscr{F} \to \mathscr{G}$ is injective/surjective then the induced map $\phi_P : \mathscr{F}_P \to \mathscr{G}_P$ is injective/surjective for every point P.

For sheaves, the situation is different. One can again form the category of sheaves of abelian groups on X, which again turns out to be an abelian category. But it might be that $\mathscr{F} \to \mathscr{G}$ is an epimorphism of sheaves without being an epimorphism of presheaves:

Proposition 2.18 (Exactness properties of sheaves). *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. Then:*

1. *If $\mathscr{F}$ and $\mathscr{G}$ are sheaves also the presheaf* $\ker(\phi)$ *is a sheaf;*
2. *If ϕ_U is injective for every open U then also $\phi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ is injective;*
3. *If $\mathscr{F}$ and $\mathscr{G}$ are sheaves then the sheafification of* $\mathrm{Im}(\phi)$ *is naturally a subsheaf of $\mathscr{G}$;*
4. *If $\mathscr{F}$ and $\mathscr{G}$ are sheaves then ϕ is an epimorphism of sheaves if and only if ϕ_P is surjective for every P.*

Proof.

1. Clearly $\ker(\phi)$ satisfies (II), so let us show that we can glue sections. Let $\bigcup_i U_i = U$ be an open cover of some open $U \subset X$ and let $s_i \in \ker(\phi_{U_i})$ be local sections. Assume that they satisfy the glueing condition. Then, since $\mathscr{F}$ is a sheaf we can at least find some $s \in \mathscr{F}(U)$ such that $s_{|U_i} = s_i$ where equality is taken inside $\mathscr{F}(U_i)$. So consider $\phi(s) \in \mathscr{G}(U)$. But $\phi(s)_{|U_i} = \phi(s_{|U_i}) = \phi(s_i) = 0$. Since $\mathscr{G}$ is a sheaf we then have $\phi(s) = 0$, i.e., $s \in \ker(\phi_U)$.
2. By construction of the sheafification, it is sufficient to show that the induced map of stalks $\phi_P : \mathscr{F}_P \to \mathscr{G}_P$ is injective for every P, which follows by the previous remark.
3. Since the map of presheaves $\mathrm{Im}(\phi) \to \mathscr{G}$ is injective by definition also $\mathrm{Im}(\phi)^+ \to \mathscr{G}^+ = \mathscr{G}$ is injective by the previous point.
4. In fact, ϕ is an epimorphism if and only if $\mathrm{Im}(\phi)^+ = \mathscr{G}$. By what we have shown, this is equivalent to $\mathrm{Im}(\phi)^+_P = \mathscr{G}_P$ for every point P. But $\mathrm{Im}(\phi)^+_P = \mathrm{Im}(\phi_P) = \mathscr{G}_P$ by assumption.

□

Now, in any abelian category, if a morphism is both an epimorphism (surjective) and a monomorphism (injective) it must be invertible. Let us verify this:

Proposition 2.19. *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphim of sheaves (not necessarily of abelian groups) such that ϕ_P is an isomorphism for every P. Then ϕ is an isomorphism.*

Proof. We know that $\phi_{|U}$ is injective for every $U \subset X$ open. We only need to show that it is also surjective. So pick $s \in \mathscr{G}(U)$ and let $P \in U$. Then we can find by assumption some $t_P \in \mathscr{F}_P$ such that $\phi_P(t_P) = s_P$. Let t_P be represented by $t'_P \in \mathscr{F}(V'_P)$ where $P \in V'_P \subset U$ is some open subset. Then the various V'_P cover X. Moreover, the various t'_P agree on the overlappings $V'_P \cap V'_Q$ due to the injectivity of ϕ, implying that they satisfy the glueing condition - hence the result. □

Returning to our discussion, the fact that surjectivity for sheaves is a local condition naturally leads to sheaves cohomology: consider a short exact sequence of abelian sheaves

$$0 \to \mathscr{F}_1 \xrightarrow{f} \mathscr{F}_2 \xrightarrow{g} \mathscr{F}_3 \to 0$$

meaning that $\mathscr{F}_1 = \ker(g)$ and $\mathscr{F}_3 = \operatorname{coker}(f)$. Then

$$0 \to \Gamma(X, \mathscr{F}_1) \xrightarrow{f_X} \Gamma(X, \mathscr{F}_2) \xrightarrow{g_X} \Gamma(X, \mathscr{F}_3)$$

is exact, but g_X need not be surjective. So $\Gamma(X, -)$ is a left-exact functor and one we study its derived functors, whose role in this case is to measure the obstructions that prevent local data from glueing into global sections. From the behaviour of such cohomology groups we then get information about the geometry of X. This will be done in the second part of the course. Let us see now a nice example which mixes complex analysis and topology:

Example 2.20. Let $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}^\times \subset \mathbb{C}$ be the circle group with its usual topology. Let $\mathscr{C}_{\mathbb{S},\mathbb{C}}$ be the sheaf of continuous functions from $\mathbb{S}$ to $\mathbb{C}$ and let $\mathscr{C}^\times_{\mathbb{S},\mathbb{C}}$ be the sheaf of continuous invertible functions from $\mathbb{S}$ to $\mathbb{C}^\times$. Then we have a short exact sequence of sheaves (verify it)

$$0 \to (2\pi i)\underline{\mathbb{Z}} \to \mathscr{C}_{\mathbb{S},\mathbb{C}} \xrightarrow{\exp} \mathscr{C}^\times_{\mathbb{S},\mathbb{C}} \to 0$$

where $\underline{\mathbb{Z}}$ is the locally constant sheaf and exp sends a local function $g : U \to \mathbb{C}$ to $\exp(g)$, where $U \subset \mathbb{S}$. Let now $f : \mathbb{S} \to \mathbb{C}^\times$ be a global section $f \in \Gamma(\mathbb{S}, \mathscr{C}^\times_{X,\mathbb{C}})$. By fixing $1 \in \mathbb{S}$ we can consider f as a continuous loop in $\mathbb{C}^\times$ with base point $f(1)$, and so f yields an element $[f] \in \pi_1(\mathbb{C}^\times, f(1)) \cong \mathbb{Z}$. Show that $f = \exp(g)$ for some global section $g : \mathbb{S} \to \mathbb{C}$ if and only if $[f] = 0$.

2.4 Functoriality of sheaves and ringed spaces

Let X, Y be topological spaces, $\mathscr{F}$ a sheaf on X, $\mathscr{G}$ a sheaf on Y and $f : X \to Y$ a continuous map.

1. Push-forward (or direct image) One defines the sheaf $f_*\mathscr{F}$ on Y by the rule $U \mapsto \mathscr{F}(f^{-1}(U))$. Show that this is a sheaf.

2. Inverse image One defines the sheaf $f^{-1}\mathscr{G}$ on X as the sheafification of $U \mapsto \mathscr{G}_{f(U)}$ (the stalk of $\mathscr{G}$ at $f(U)$).

Note that although f_* is easier to define, it is easier to compute the stalks of f^{-1}:

Proposition 2.21. *For $x \in X$ we have natural isomorphisms:*

1. $(f_*\mathscr{F})_{f(x)} \cong \mathscr{F}_{f^{-1}(f(x))}$.

2. $(f^{-1}\mathscr{G})_x = \mathscr{G}_{f(x)}$.

Proof. This follows directly from the definition. □

In particular, f^{-1} is exact and f_* is left-exact. We shall define f^* later in the context of schemes. Finally, suppose we are in the situation of the previous definition, and that we want to define a morphism of sheaves $\mathscr{G} \to \mathscr{F}$ 'along f'. We can then either consider $\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})$ or $\mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$: the two are the same.

Proposition 2.22 (adjunction inverse/direct image). *There is a natural isomorphism* $\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$.

Proof. We construct maps in both directions. Take $\phi \in \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$, so for every open $V \subset Y$ we have a compatible set of morphisms $\phi_V : \mathscr{G}(V) \to \mathscr{F}(f^{-1}(V))$. Pick any open $U \subset X$ and suppose that $U \subset f^{-1}(V)$ (which is equivalent to $f(U) \subset V$). Then we get a map

$$\mathscr{G}(V) \xrightarrow{\phi_V} \mathscr{F}(f^{-1}(V)) \xrightarrow{\rho_{f^{-1}(V),U}} \mathscr{F}(U)$$

and this is easily seen to define a morphism

$$(f^{-1}\mathscr{G})(U) = \varinjlim_{f(U)\subset V} \mathscr{G}(V) \to \mathscr{F}(U)$$

and thus an element of $\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})$. To obtain a map in the other direction, pick $\psi \in \mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})$ and let $V \subset Y$ be any open subset. Put $U = f^{-1}(V)$, which is open in X. So by definition of direct limit we have a map $\mathscr{G}(V) \to f^{-1}(\mathscr{G})(U)$ and hence we obtain a morphism

$$\mathscr{G}(V) \to f^{-1}(\mathscr{G})(U) \xrightarrow{\psi_U} \mathscr{F}(U) = \mathscr{F}(f^{-1}(V)) = (f_*\mathscr{F})(V)$$

hence an element of $\mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$. One can check that these are one the inverse of the others. □

For example, let $X \xrightarrow{f} Y$ be a morphism of algebraic varieties (or any example in point (1) from 2.1.1); then for any open $V \subset Y$ and any regular function $h: V \to K$ the composition $f^{-1}(V) \xrightarrow{f} V \xrightarrow{g} K$ is a regular function on $f^{-1}(V) \subset X$, and so we obtain a morphism of sheaves of rings $f^{\#}: \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Remark 2.23. Let X, Y be differentiable (or complex) manifolds, and let $f: X \to Y$ be a continuous morphism. Then we get a morphism as before $f^{\#}: \mathcal{C}_Y \to f_*\mathcal{C}_X$. Since $\mathcal{C}_X^\infty \subset \mathcal{C}_X$ is a subsheaf and f_* is left-exact, also $f_*\mathcal{C}_X^\infty \subset f_*\mathcal{C}_X$ (similarly, $f_*\mathcal{O}_X^{\text{hol}} \subset f_*\mathcal{C}_X$)). Show that f is differentiable resp. holomorphic if and only if $f^{\#}(\mathcal{C}_Y^\infty) \subset f_*(\mathcal{C}_X^\infty)$ resp. $f^{\#}(\mathcal{O}_Y^{\text{hol}}) \subset f_*(\mathcal{O}_X^{\text{hol}})$.

This means that the sheaves $\mathcal{C}_X^\infty$ or $\mathcal{O}_X^{\text{hol}}$ carry all the information that turn X into a differentialble or complex manifold. For example, we can substitute the (cumberstone) notion of atlas with the one of sheaf.

Definition 2.24. A ringed space $(X, \mathcal{O}_X)$ is a topological space X together with a sheaf of rings. We say that $(X, \mathcal{O}_X)$ is a locally ringed space if the stalk of $\mathcal{O}_X$ at any point $x \in X$ is a local ring.

A morphism $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is a pair $(f, f^{\#})$ where $f: X \to Y$ is a continuous map and

$$f^{\#} \in \mathrm{Hom}_X(f^{-1}\mathcal{O}_Y, \mathcal{O}_X) = \mathrm{Hom}_Y(\mathcal{O}_Y, f_*\mathcal{O}_X)$$

is a morphism of sheaves of rings satisfying: for every $x \in X$, the induced map on stalks

$$\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,y}$$

is a morphism of local rings (meaning that sends the maximal ideal to the maximal ideal).

For example, for any map of rings $A \to B$ and any prime $\mathfrak{p} \subset B$ the induced map $A_{\mathfrak{p}^c} \to B_{\mathfrak{p}}$ is local by the definition of the contraction $\mathfrak{p}^c$. This shows that morphisms of algebraic varieties are automatically local. On the other hand, note that the inclusion of $K[[t]]$ in its fraction field $K((T))$ is not a map of local rings.

Definition 2.25. An algebraic variety over K is a locally ringed space $(X, \mathcal{O}_X)$ which is covered by open subsets U_i such that $(U_i, \mathcal{O}_{|U_i})$ is isomorphic, as a locally ringed space, to an affine K-variety (here, $\mathcal{F}_{|U}$ denotes the inverse image of $\mathcal{F}$ along the open immersion $U \subset X$).

For example, let $(X, \mathcal{O}_X)$ be an affine algebraic variety and let $U \subset X$ be an open subset. Then $(U, \mathcal{O}_{|U})$ is an algebraic variety. It is called a *quasi-affine variety.*

Remark 2.26. Quasi-affine varieties need not be affine. The classical example is $U = K^2 \setminus 0$ with the induced sheaf of functions. A regular function $U \to K$ is in particular a rational function $h = f/g \in K(x, y)$ with $f, g \in K[x, y]$ coprime. Assume that g is not a unit. We claim that $V(g) \not\subset V(f)$ inside K^2. Let $\prod_i \pi_i^{\alpha_i}$ be a prime decomposition of f and $\prod_j \pi_j'^{\alpha_i'}$ be one of g. Then $\pi_i \neq \pi_j'$ for any i, j. Suppose $V(g) \subset V(f)$ and fix a prime π dividing g. Then $V(\pi)$ is an irreducible variety of dimension one. Since $V(\pi) \subset \bigcup_j V(\pi_j')$ and each $V(\pi_j')$ is irreducible of dimension one, we have that $V(\pi) = V(\pi')$ for some π' dividing f. But this means that $\pi = \pi'$ up to units by Nullstellensatz, which is a contradiction to coprimality. So $V(\pi) \neq V(\pi_j')$ for every j. Now $V(\pi_j') \cap V(\pi)$ is closed of $V(\pi)$, hence it must be a bunch of points (for it cannot be the whole thing). Hence up to finitely many points none of the points in $V(\pi)$ are in $V(f)$. Since $V(\pi)$ contains infinitely many points we conclude that there must be some $u \in K^2 \setminus 0$ such that $g(u) = 0$ but $f(u) \neq 0$, showing that f/g is not regular at u.

Another proof: assume that $h \in \cap_{\mathfrak{m} \neq (x_0, x_1, x_2)} K[x, y]_{\mathfrak{m}} \subset K(x, y)$. Using the same trick as in point (3) of Theorem 2.10 we see that $I = \{g \in K[x, y] : \ gh \in K[x, y]\}$ is an ideal such that $\sqrt{I} = (x_0, x_1, x_2)$. Hence $(x_0, x_1, x_2)^k \subset I$ for some $k > 0$ because (x_0, x_1, x_2) is maximal. This means that $x_i^k h \in K[x, y]$ for every i. Now use that $K[x, y]$ is a UFD and find the contradiction as before.

We will show later in the course (but you can try to prove it yourself now) that for affine varieties $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ any morphism of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is induced by a unique morphism of varieties, hence by a unique map of K-algebras $A(Y) = \Gamma(Y, \mathcal{O}_Y) \to A(X) = \Gamma(Y, \mathcal{O}_X)$.

3 Lecture III / IV

The projective space $\mathbb{P}^n(K)$ is the simplest and most useful compactification of the affine spaces K^n. Originally, it was discovered by painters to study prospective and its use in algebraic geometry was initiated much later, that is, when the need to work with complete (or proper or 'compact') varieties became apparent. We prefer to work with such spaces for many reasons, most notably: finitely generation of cohomology, existence of intersection theory e.g., Bézout's theorem, Poincaré duality, existence of limits, etc).

The projective plane has its origins in practical problems, such as the mathematical understanding of perspective, and this motivates its abstract definition in a precise way which is not so hard to illustrate.

Consider the (x, y)-plane $P \subset \mathbb{R}^3$, with an observer standing upright on it: their feet are at the origin $O = (0, 0, 0)$ and their head is at the point $H = (0, 0, 1)$ (up to rescaling). On the whole plane P there is a picture which the observer views from H. As the observer looks farther across P, the picture appears increasingly compressed toward the horizon and when its line of sight becomes parallel to P, then the observer is looking directly at the *horizon* and raising the gaze further above only reveals empty space. Thus, although P is infinite, from the point of view of the observer it appears to bounded by the horizon. For example consider an infinite railway drawn on P, with O at its midpoint. Looking straight down at your feet the two rails appear as parallel lines. As you lift your gaze along the direction of the railway, the rails seem to converge until they appear to meet at the horizon. The horizon, however, is not a physical object but rather a visual effect, so nothing can really meet there. In projective geometry we incorporate the horizon in a mathematical way to P and thus obtain a new space, the *projective plane*. In this space the two lines will then really intersect on the horizon, also called the *line at infinity*.

This also gives an example of a geometry where Euclid's fifth postulate does not hold.

3.1 Projective spaces

Let K be as usual an algebraically closed field.

Definition 3.1. The n-th dimensional projective space is defined as

$$\mathbb{P}^n(K) := (K^{n+1} \setminus 0)/K^\times$$

where $K^\times$ acts on $K^{n+1} \setminus 0$ via

$$\lambda \cdot (x_0, \dots, x_n) \mapsto (\lambda x_0, \dots, \lambda x_n).$$

One denotes $[x_0 : \ldots : x_n]$ the point corresponding to $(x_0, \ldots, x_n) \in K^{n+1} \setminus 0$. Similalrly, if W is any K-vector space we let $\mathbb{P}(W) = (W \setminus 0)/K^\times$.

Thus $\mathbb{P}(W)$ is in natural bijection with the set of lines $L \subset W$ passing through the origin (which should be interpreted as the point H if we keep the analogy from the introduction, i.e., each line corresponds to a point in your view).

Remark 3.2. If $K = \mathbb{C}$ and $\mathbb{D}^n = \{\mathbf{z} \in \mathbb{C}^{n+1} : |\mathbf{z}| = 1\}$ then the induced map $\mathbb{D}^n \to \mathbb{P}^n(\mathbb{C})$ is surjective, which shows in particular that $\mathbb{P}^n(\mathbb{C})$ is compact when equipped with the analytic topology.

Once we define properness for schemes, we will prove that $\mathbb{P}^n$ is compact by purely algebraic methods.

3.1.1 Zariski topology on $\mathbb{P}^n(K)$

Consider $K^{n+1} \setminus 0 \subset K^{n+1}$ with the induced Zariski topology, and denote by $\pi : K^{n+1} \setminus 0 \to \mathbb{P}^n(K)$ the quotient map. We endow $\mathbb{P}^n(K)$ with the quotient topology, so that $Z \subset \mathbb{P}^n(K)$ is closed if and only if $\pi^{-1}(Z) \subset K^{n+1} \setminus 0$ is closed. This means that there is a unique Zariski closed $C(Z) \subset K^{n+1}$ such that $C(Z) \setminus 0 = \pi^{-1}(Z)$. So $C(Z) = V(I)$ for some ideal $I \subset K[x_0, \cdots, x_n]$. By construction, if $x \in C(Z)$ and $k \in K$ then $k \cdot x \in C(Z)$ too: that is, if $x \in C(Z) \setminus 0$ then also the line L_x joining x to 0 is contained in $C(Z)$. Such closed subsets are called cones (with vertex 0 and base Z). So what kind of ideals $I \subset K[x_0, \cdots, x_n]$ yield cones as vanishing locus? One can figure out the answer easily 'by hand'. More conceptually, for any $k \in K^\times$ consider the automorphism of rings $\lambda_k : K[x_0, \cdots, x_n] \to K[x_0, \cdots, x_n]$ sending $x_i \mapsto kx_i$. This defines an action of $K^\times$ on $K[x_0, \cdots, x_n]$. So $V(I)$ is a cone if and only if $\lambda_k(I) = I$ for every $k \in K^\times$. We decompose now the polynomial ring into eigenspaces for this action:

$$K[x_0, \cdots, x_n] = \bigoplus_{d \geq 0} K[x_0, \cdots, x_n]_d$$

where $K[x_0, \cdots, x_n]_d$ consists of the K-vector space of homogeneous polynomials of degree d, i.e.,

$$K[x_0, \cdots, x_n]_d = \{P \in K[x_0, \cdots, x_n] : \lambda_k(P) = k^d P \text{ for every } k \in K^\times\}.$$

Proposition 3.3. *An ideal $I \subset K[x_0, \cdots, x_n]$ is invariant for the action of $K^\times$ if and only if it is generated by homogeneous polynomials.*

Such ideals are called *homogeneous* ideals.

Proof. If I is generated by homogenous polynomials, then it is clearly invariant under the action of $K^\times$. Now, assume that I is invariant, pick $f \in I$ and decompose $f = \sum_d f_d$ into homogeneous components. We claim that each $f_d \in I$ as well. We prove this by induction on the number $N = |\{d : f_d \neq 0\}|$. So if $N = 1$ there is nothing to prove. If $N > 1$ let d_0 be the maximum of $\{d : f_d \neq 0\}$ and let $\mu \in K$ be a primitive d_0-root of unity. Then $f - \delta_{\mathfrak{m}} u(f) \in I$ by assumption and

$$f - \lambda_{\mathfrak{m}} u(f) = \sum_{d<d_0} (1 - \mu^d) f_d$$

Since μ is primitive $1 - \mu^d \neq 0$ for any $0 < d < d_0$ hence by induction each $f_d \in I$ for $d < d_0$ and so also $f_d \in I$. But this implies that the ideal I is generated by the homogeneous components of its elements, which means that I a homogeneous ideal. □

Thus every homogeneous ideal gives a closed subset of $\mathbb{P}^n(K)$, with one small caveat. Note that the action of $K^\times$ extends naturally to an action on K^{n+1} and $K^{n+1}/K^\times = \{0\} \sqcup \mathbb{P}^n(K)$. The homogeneous ideal $(x_0, \dots, x_n)$ then would correspond to the point $\{0\}$ of $K^{n+1}/K^\times$. Since we are throwing this point away, we should consider only homogeneous ideals which do not contain the ideal $(x_0, \dots, x_n)$, which is called the *irrelevant ideal*. Note in particular that $V(x_0, \cdots, x_n) = \emptyset$. One can prove in this way a correspondence analogue to the one from affine varieties:

Proposition 3.4. *Let $X = V(\mathfrak{p})$ be a projective variety and let $R = K[x_0, \dots, x_n]/\mathfrak{p}$ which is a graded ring in a natural way. Let R_+ be the ideal generated by homogeneous elements of positive degree. Then there is a one-to-one correspondence between closed subsets of X and radical homogeneous ideals of R strictly contained in R_+.*

Example 3.5. Take $F \in K[x_0, \cdots, x_n]_d$ for $d \geq 1$, and consider the homogeneous ideal $I = (F)$. Note that $F(kx_0, \cdots, kx_n) = k^d F(x_0, \cdots, x_n)$ hence the set

$$\{[x_0 : \ \dots, \ : \ x_n] \in \mathbb{P}^n(K) : \ F(x_0, \cdots, x_n) = 0\}$$

is a well-defined projective hypersurface.

An irreducible closed subset of $\mathbb{P}^n(K)$ is called a projective variety.

Natural open cover To describe the geometry of $\mathbb{P}^n(K)$ and of projective varieties in general, we turn the introduction into a mathematical construction. Consider any hyperplane $W \subset K^{n+1}$ (i.e. a linear subspace of dimension n) not passing through the origin. Every point $w \in W$ defines a unique line

$L_w \subset K^{n+1}$ passing through the origin and w, hence a unique point $\phi_W(w) \in \mathbb{P}^n(K)$. If $\overline{W}$ denotes the hyperplane parallel to W and passing through 0 (the directions at the horizon) then the only points of $\mathbb{P}^n(K)$ which are not in the image of ϕ_W are those lines inside $\overline{W}$, that is

$$\mathbb{P}^n(K) \setminus \mathrm{Im}(\phi_W) = \mathbb{P}(\overline{W}) \cong \mathbb{P}^{n-1}(K).$$

This means that the horizon is in fact a projective space of dimension one smaller. By repeating this yields decompositions

$$\mathbb{P}^1(K) \cong K \sqcup \{*\}$$

$$\mathbb{P}^2(K) \cong K^2 \sqcup \mathbb{P}^1(K) \cong K^2 \sqcup K \sqcup \{*\}$$

and so on. So in the projective line, we add only one point to K to obtain $\mathbb{P}^1(K)$, which is the infinity. In the projective plane we need to add all the directions parallel to the affine plane, which corresponds to the intuitive idea of horizon.

Proposition 3.6. *Let $W_0, \cdots, W_n \subset K^{n+1} \setminus 0$ be linearly independent hyperplanes $\overline{W}_0 \cap \cdots \cap \overline{W}_n = 0$. Then*

$$\bigcup \mathrm{Im}(\phi_{W_i}) = \mathbb{P}^n(K).$$

So every point of $\mathbb{P}^n(K)$ lies in the image of some ϕ_W; moreover, the map $\phi_W : W \to \mathbb{P}^n(K)$ will soon turn out to be an open immersion. In this way we get a local description of $\mathbb{P}^n(K)$ around every point: $\mathbb{P}^n(K)$ is covered by open subsets which are also affine algebraic varieties (in our case, K^n). In particular, it is an algebraic variety.

3.2 Functions on $\mathbb{P}^n_K$

If G is a group acting on some space X and $Y = X/G$ exists, then we expect a function on Y to be a function on X which is invariant under the action of G. Since for $n \geq 1$ the regular functions on $K^{n+1} \setminus 0$ are the same as the regular functions on K^{n+1} (why?) a regular function $\mathbb{P}^n(K) \to K$ should then correspond to a polynomial in $K[x_0, \dots, x_n]$ which is invariant under the action of $K^\times$, i.e., a constant polynomial. This heuristic argument shows an important feature of projective (or complete) varieties: any morphism to an affine variety must be constant.

On the other hand, let F be a homogeneous polynomial, and let $D_+(F) \subset \mathbb{P}^n(K)$ be the corresponding principal open subset. Now $\pi^{-1}(D_+(F)) = D_F \setminus 0 \subset K^{n+1} \setminus 0$ and $\mathcal{O}(D_F) = K[x_0, \dots, x_n]_F$. Using a similar reasoning, the regular functions $D_+(F) \to K$ should correspond to the fractions in $K[x_0, \dots, x_n]_F$

which are invariant under the action of $K^\times$. Note that we can extend the degree function on the whole $\mathrm{Frac}(K[x_0, \dots, x_n])$, so the invariant functions correspond to the degree zero part of $K[x_0, \dots, x_n]_F$, which is a subring denoted by $K[x_0, \dots, x_n]_{(F)}$. Then every element $G/F^k \in K[x_0, \dots, x_n]_{(F)}$ where G is homogeneous of degree $k \deg(F)$ describes a regular function on $D_+(F)$ via the rule

$$D_+(F) \ni [x_0 : \ \dots \ : \ x_n] \mapsto \mathrm{Frac}G(x_0, \dots, x_n)F(x_0, \dots, x_n)^k$$

which is indeed a well-defined function.

Now, note that the open subsets $D_+(F)$ form a basis for the Zariski topology, more precisely: any open subset is the finite union of prinicipal open subsets (and any finite intersection of principal open subsets is again principal).

Excercise 3.7. Let X be a topological space and let $\{U_i\}_{i \in I}$ be a basis for the topology for X, as above. Assume that for any $i \in I$ we have a group $\tilde{\mathscr{F}}(U_i)$ together with a compatible system of restriction maps. Show that the same construction of the sheafification yields a unique sheaf $\mathscr{F}$ on X.

In this way, we can define the sheaf of rings $\mathscr{O}_{\mathbb{P}^n}$. Similarly, if $\mathfrak{p} \subset K[x_0, \dots, x_n]$ is a homogeneous prime ideal and $R = K[x_0, \dots, x_n]/\mathfrak{p}$ is the associated graded algebra, we define the sheaf $\mathscr{O}_X$ on $X = V(\mathfrak{p})$ by declaring $\mathscr{O}_X(D_+(F)) = R_{(F)}$ for every $F \in R$. The pair $(X, \mathscr{O}_X)$ is then called a projective variety.

3.2.1 Projective varieties are locally affine algebraic varieties

Let now $\mathfrak{p} \subset K[x_0, \cdots, x_n]$ be a homogeneous ideal and let $X = V(\mathfrak{p})$ be the associated projective variety. Let $R = K[x_0, \cdots, x_n]/\mathfrak{p}$ which is a graded ring. Fix a homogeneous element $F \in R$ (of some degree $d \geq 1$) and let $R_{(F)}$ denote the degree−zero part of the graded ring R_F. We will prove that the open subset $D_+(F) := X \setminus V(F) = \{x \in X \mid F(x) \neq 0\}$ is (functorially) isomorphic to the set of maximal ideals of $R_{(F)}$:

$$D_+(F) \cong \ \mathrm{mSpec}\left(R_{(F)}\right).$$

Note that $R_{(F)}$ is a finitely generated K-algebra, so $\mathrm{mSpec}\left(R_{(F)}\right)$ is an affine algebraic varieties.

- If $x = [x_0, \dots, x_n] \in D_+(F)$ then $F(x) \neq 0$ and so we have a well-defined evaluation map $R_F \to K$ evaluating each rational function at $(x_0, \dots, x_n)$. We restrict this map to the degree zero part and obtain $\mathrm{ev}_x : R_{(F)} \to K$; notice that this is well-defined (i.e., does not depend on the chosen representative of $(x_0, \dots, x_n)$ of x). The kernel of ev_x is then a maximal ideal $\mathrm{m}_x \in \mathrm{mSpec}\left(R_{(F)}\right)$, which defines the map $D_+(F) \to \mathrm{mSpec}(R_{(F)})$.

- Let now $\mathfrak{m} \in \mathrm{mSpec}\left(R_{(F)}\right)$ be a maximal ideal. The extension $\mathfrak{m}^e$ of $\mathfrak{m}$ to R_F is then $\bigoplus_{d\in\mathbb{Z}} F^d\mathfrak{m}$. Choose any $\lambda \in K^\times$ and consider the unique ring map $R_F \to K$ sending $R_{(F)}$ to $R_{(F)}/\mathfrak{m} = K$ and F to λ. This defines a maximal ideal of R_F hence a point $(x_0, \dots, x_n) \in K^{n+1}$. This cannot be 0 because $F(x_0, \dots, x_n) = \lambda \neq 0$ and therefore defines a point $[x_0 : \dots : x_n] \in D_+(F)$. Different choices of λ yield the same point.

By contruction, these are one the inverse of the others. As we shall prove later in more generality the map can be upgraded to an isomorphism of ringed spaces $(D_+(F), \mathcal{O}_{|D_+(F)}) \cong (\mathrm{mSpec}(R_{(F)}), \mathcal{O}_{\mathrm{mSpec}(R_{(F)})})$. In particular:

Corollary 3.8. *Every principal open subset of a projective variety is an affine variety in a natural way.*

Example 3.9.

- Let $W \subset K^{n+1}$ and $\overline{W}$ be as before. Then $\overline{W}$ is the zero locus of a homogeneous degree one polynomial F. Let us compute $K[x_0, \cdots, x_n]_{(F)}$. We can assume that $F = x_i$ for some i; then $K[x_0, \cdots, x_n]_{(x_i)} = K[x_0/x_i, \dots, x_n/x_i]$ which is the polynomial ring in the n-variables x_j/x_i for $j \neq i$. Show that $W \cong \mathrm{mSpec}(K[x_0, \dots, x_n]_{(F)})$ in a natural way and that the $\phi_W : W \to \mathbb{P}^n(K)$ corresponds to the identification

$$\mathrm{mSpec}(K[x_0, \cdots, x_n]_{(F)}) \xrightarrow{\sim} D_+(F)$$

from before.

- Let $W_i \subset K^{n+1}$ be the hyperplanes $V(x_i - 1)$ so that $\overline{W}_i = V(x_i)$. Let $X \subset \mathbb{P}^n(K)$ be a subvariety. To understand X we shall 'project it' to the hyperplanes W_i: Let $D_+(x) \cap X$ be the principal open subset corresponding to the restriction of x_i to X. If this is empty, then $X \subset V(x_i) \cong \mathbb{P}^{n-1}$ so we can assume that $D_+(x_{i|X}) \neq \emptyset$ up to moving to a smaller ambient space. The restriction of $D_+(x_i) \xrightarrow{\cong} W_i$ to X then identifies $D_+(x) \cap X$ with an algebraic variety $X_i \subset W_i$. If we understand all the algebraic varieties X_i then we can recover X by glueing them over their intersections, since the varieties $X_i \subset X$ form an open cover of X.

  For example, let $F \in K[x_0, \dots, x_n]$ be a homogeneous polynomial of degree d. Define $F_i = F/x_i^d \in K[x_0, \dots, x_n]_{(x_i)} \cong K[\dots x_j/x_i \dots]$. So F_i is a polynomial of degree d obtained by 'dehomogenizing' F at its i-th coordinate. If $X = V(F)$ then $X_i = V(F_i) \subset W_i \cong K^n$. This is very useful for concrete computations. If $X = V(\mathfrak{p})$ then to obtain X_i one dehomogenise the generators of $\mathfrak{p}$ at the i-th place and consider the ideals $\mathfrak{p}_i \subset K[\dots x_j/x_i \dots]$ generated by those.

3.2.2 Morphisms to affine varieties

We now prove the following:

Theorem 3.10. *Let $(X, \mathcal{O}_X)$ be a projective variety and let $(Y, \mathcal{O}_Y)$ be an affine variety. Then any morphism $f : X \to Y$ is constant.*

Proof. It is enough to show that any morphism $X \to K$ is constant, that is, it is enough to show that $\Gamma(X, \mathcal{O}_X) = K$. Let $X = V(\mathfrak{p})$ and let $R = K[x_0, \dots, x_n]/\mathfrak{p}$ be the corresponding graded ring. We denote by F the fraction field of R and by F_0 its degree-zero part. For every Zariski open $U \subset X$ we have a natural inclusion $\Gamma(U, \mathcal{O}_{|U}) \subset F_0$ and therefore we can treat all these rings as subrings of the same field. Now, for any $F \in R$ homogeneous of degree $d > 0$ we have $\Gamma(D_+(F), \mathcal{O}) = R_{(F)} \subset F_0$ and therefore $\Gamma(X, \mathcal{O}_X) = \bigcap_F R_{(F)}$. Now note that $x_1, \cdots, x_n$ generate R as a graded ring (we can throw away the $x_i \in K$). So for every $g \in \Gamma(X, \mathcal{O}_X)$ there must be $N > 0$ such that for $x_i^N g \in R_N$. Now we use a trick (see Hartshorne): replace N with $N(n+1)$ and consider a monomial in $x_1, \cdots, x_n$ of degree $N(n+1)$. Then at least one x_i must appear with power $\geq N$ and therefore $f R_{N(n+1)} \subset R_{N(n+1)}$. This also implies $f^k R_{N(n+1)} \subset R_{N(n+1)}$ for every $k \geq 0$. But then also $f^k R_M \subset R_M$ for every $M \geq N(n+1)$.

Consider now the graded ring $R[f] \subset F$. We have an inclusion of R-modules (forget the gradings)

$$R \subset R[f] \subset x_i^{-N} R \subset F$$

for some $x_i \neq 0$; but R is Noetherian and $x_i^{-N} R$ is obviously finitely generated R-module; so also the submodule $R[f]$ must be finitely generated, i.e., f is integral over R. If R were integrally closed (which means that the variety is normal) then we would conclude that $f \in R$ hence $f \in R_0 = K$. In general, write an equation $f^m + a_{m-1} f^{m-1} + \cdots + a_0 = 0$ with $a_i \in R$. Finally, considering the degree zero part of the equation we can assume $a_i \in K$ and, finally, $f \in K$ since K is algebraically closed. □

If we knew that X were proper (i.e. complete), then we could also prove the result as follows: let $f : X \to K$ by a function and consider the induced function $f : X \to K \subset \mathbb{P}^1(K)$. Since X is proper $f(X) \subset \mathbb{P}^1(K)$ is closed. But the only closed subsets of $\mathbb{P}^1(K)$ which are contained in the principal open subset K are the finite sets. Since X is connected, $f(X)$ must be a point.

Remark 3.11. In complex geometry this is a manifestation of the maximum modulus principle. That is, let X be a compact complex manifold and let $f : X \to \mathbb{C}$ be a holomorphic function. Since X is compact there must be a point $x \in X$ such that $|f(x)|$ achieve its maximum. Then take a small open subset $x \in U \subset X$. By the maximum module principle, $f_{|U}$ cannot achieve its maximum in U unless f is constant.

Let us show how the proof works in the easiest example. Consider $X = \mathbb{P}^1(K)$. Then we can cover X with the two open subsets $U_0, U_1 \cong K$. Now a function $f_0 : U_0 \to K$ is represented by a polynomial $P(x_1/x_0)$ and similarly a function $f_1 : U_1 \to K$ is represented by a polynomial $Q(x_0/x_1)$. In practice, this means that $f([x_0, x_1]) = P(x_1/x_0)$ if $x_0 \neq 0$ and $f([x_0, x_1]) = Q(x_0/x_1)$ if $x_1 \neq 0$. The compatibility condition simply means that $P(x_1/x_0) = Q(x_0/x_1)$ if both $x_0, x_1 \neq 0$. By putting $t = x_1/x_0$ we then have $P(t) = Q(t^{-1})$ which immediately shows that both P and Q are constant.

Corollary 3.12. *Let $Z \subset \mathbb{P}^n(K)$ be a positive dimensional projective variety and let F be homogeneous of degree $d \geq 1$. Then $V(F) \cap Z \neq \emptyset$.*

Proof. If $V(F) \cap Z = \emptyset$ then $Z \subset D_+(F)$ which is affine. Hence Z must be a point. □

Alert on graded rings We cannot expect the same dictionary between finitely generated K-algebras and affine varieties to work also for graded rings, and it is worthwhile to study the failure of this functoriality. For simplicity we assume that all graded rings are generated in degree one, as it happens for the graded quotients of $K[x_0, \ldots, x_n]$. A morphism $f : R \to S$ of degree d is by definition a morphism of rings such that $f(R_n) \subset S_{nd}$. For example, $\phi : K[y_0, \ldots, y_m] \to K[x_1, \ldots, x_n]$ has degree d if and only if $f(y_i) = F_i$ is homogeneous of degree d for every i. We also define $R(d)$ as the graded ring $\oplus_n R_{nd}$ which comes with a natural inclusion $R(d) \subset R$ which has degree d. Note that $R(d)$ also is generated in degree one.

Now, one reason why such a morphism does not induce a map on the corresponding projective varieties is only due to the fact that certain maximal ideals may be sent to the irrelevant ideal. Geometrically, every such ϕ induces a map of affine cones $C(X) \to C(Y)$ and it may happen that some rays of $C(X)$ are collapsed to the origin, and so the correponding points of X cannot have a well-defined image. Now, in mSpec(S) and Spec(S) the ideals restricting to R_+ form a closed subset which corresponds to $V(R_+^e)$ due to the following lemma:

Lemma 3.13. *If $\phi : A \to B$ is a map of rings and I is an ideal of A then for a prime ideal $\mathfrak{p} \subset B$ we $I \subset \mathfrak{p}^c$ if and only if $I^e \subset \mathfrak{p}$.*

Proof. If $I \subset \mathfrak{p}^c$ then $I^e \subset (\mathfrak{p}^c)^e \subset \mathfrak{p}$. If $I^e \subset \mathfrak{p}$ then $I \subset (I^e)^c \subset \mathfrak{p}^c$. □

For example, if $F_1, \cdots F_m$ are homogeneous polynomials of degree d in n variables they should give an induced morphism $\mathbb{P}^n(K) \setminus V(F_1, \ldots, F_m) \to \mathbb{P}^m(K)$. We make this a statement:

Proposition 3.14. *Any map $f : R \to S$ as before induces a morphism of algebraic varieties $U \to Y$ where $U = X \setminus V(R_+^e)$ with the induced sheaves of functions.*

On the other hand, the fact that $V(R^e_+) \neq \emptyset$ does not automatically imply that the map above does not extend to the whole X:

Example 3.15. Let $F, G \in K[x_0, x_1, x_2]$ be polynomials of degree d and consider the induced map ϕ given by $x = [x_0 : x_1 : x_2] \mapsto [F(x) : G(x)]$. This is not defined in the common zeros of F and G, which is non-empty by the previous corollary. Let $x \in V(F) \cap V(G)$ and let $L \subset \mathbb{P}^2$ be any line passing through x. Then the restriction $\phi_{|L-\{x\}}$ extends to a morphism $\phi_L : L \to \mathbb{P}^1(K)$ (see the next section or prove it). Consider now the set of lines $\mathscr{L}_x = \{L \subset \mathbb{P}^2(K) : x \in L\}$. Note that $\mathscr{L}_x \cong \mathbb{P}^1(K)$. Thus for every $L \in \mathscr{L}_x$ we can associate a value $\phi_{|L}(x) \in \mathbb{P}^1(X)$. Show that this defines a morphism $\mathscr{L}_x \to \mathbb{P}^1(K)$ and determine it explicitly.

It is more difficult to construct examples when L is replaced by a higher dimensional variety.

3.3 Compactification of affine varieties

Projective spaces allow us to compactify affine varieties very easily. This may lead to singular projective varieties (where the singularities happen at infinity if your original variety was non-singular) but it is easy to check its behaviour at infinity.

Let $V \cong K^n$ and let $X \subset V$ be an affine variety and consider the morphism $V \to K^n \oplus K$ sending $v \mapsto (1, v)$. The image of this is then the hyperplane $W_0 = (x_0 - 1)$ and so X can be seen as a Zariski closed subset of $W_0 \cong D_+(x_0) \subset \mathbb{P}^n(\mathbb{K})$. Its Zariski closure $\overline{X} \subset \mathbb{P}^n(\mathbb{K})$ is then the sought for compactification of X.

To understand this, let us again look at the case of hypersurfaces $X = V(f)$ where $f \in K[x_1, \cdots, x_n]$ has degree d. We let $F = x_0^d f(x_1/x_0, \cdots, x_n/x_0)$ This is now homogeneous and $\overline{X} = V(F)$. For example let $\mathbb{P}^n(\mathbb{K}) \setminus D_+(x_0) = \mathbb{P}(W)$ be the hyperplane at infinity. Then if f_d is the homogeneous part of f of maximal degree we have

$$\overline{X} \cap (\mathbb{P}^n(\mathbb{K}) \setminus D_+(x_0)) = V(f_d) \subset \mathbb{P}(W).$$

Let us use this equation in some examples:

Example 3.16. 1. (Lines) This is again a verification of the introduction: see $\mathbb{P}^2(K)$ as the compactification of K^2 and take two lines $V(ax + by + c)$ and $V(a'x + b'y + c')$. Then these lines meet the line at infinity, which is isomorphic to $\mathbb{P}(K^2) = \mathbb{P}^1(K)$, respectively at $[a : b]$ and $[a' : b']$. But these are the same points if and only if the lines are parallel.

2. (Conics) Recall that an affine conic $C \subset K^2$ is the zero set of an irreducible quadratic polynomial $q(x, y) = q_2(x, y) + q_1(x, y) + c$. If we complete the affine plane as before and let $\overline{C}$ be the closure of C in $\mathbb{P}^2(K)$, then the intersection of C at infinity is the zero locus in $\mathbb{P}^1(K)$ of $q_2(x, y)$. This can either consists of two different points or of one point with multiplicity two (in the right affine chart, this is nothing but a quadratic equation in one variable). The frist case happens when C is an hyperbola and the second when the line at infinity is tangent to $\overline{C}$) i.e., C is a parabola. If $K = \mathbb{R}$ there is another possibility, namely, the points of $V(q_2)$ are not real (e.g. $q_2 = x^2 + y^2$). In this case the zero locus of f in $\mathbb{R}^2$ does not intersect the line at infinity and so C must be compact, i.e., an ellipse.

3. (Easy Bezout) Let $F \in K[x_0, x_1, x_2]$ be homogeneous of degree $d > 0$ and let $L \subset \mathbb{P}^2(K)$ be any line not contained in $V(F)$ (this is automatically if F does not contain a linear factor). Prove that $L \cap V(F)$ consists precisely of d-points when counted with multiplicity (reduce the statement to the fundamental theorem of algebra).

4. (Conics are lines) We now show that if $F \in K[x_0, x_1, x_2]$ is a homogeneous irreducible polynomial of degree 2 the associated conic $C = V(F) \subset \mathbb{P}^2(K)$ is actually isomorphic to $\mathbb{P}^1(K)$. Take any point $P \in C$ and take any line $L \subset \mathbb{P}^1(K)$ (i.e, the zero set of a linear homogeneous equation) with $P \notin L$. For any $c \in C \setminus P$ let L_c be the unique line passing through P and c. Since $P \notin L$ we have that $L \neq L_c$ hence $L \cap L_c$ consists precisely of one point $\phi(c)$. Show that this defines a map $C \setminus P \to L \cong \mathbb{P}^1(K)$ which extends to an isomorphism $C \cong \mathbb{P}^1(K)$ (construct the inverse; note that you can choose L in any nice position). Compare this with the conic examples from the previous lecture.

3.3.1 Glimpse of properness

Let X be a compact topological space and let

$$f : [0, 1) \to X$$

be a continuous function. If f is well-behaved around 1 (e.g. if $f(a_n)$ is Cauchy for every $a_n \in [0, 1)$ converging to 1) then the limit $\lim_{x \to 1} f(x)$ must exist in X. In algebraic geometry we do not have open intevervals, and the best we can do is to replace them with one dimensional open subsets. For instance, we can replace $[0, 1)$ with $U = K \setminus 0$ and consider a morphism $f : U \to X$ where X is some projective variety. We then ask: what does it mean that f is well-behaved around the missing point 0, and if well-behaved, does the limit always exist? It turns out that any such morphism is automatically well-behaved (the reason for this is that rational functions are meromorphic functions with poles as singularities. For instance the function $(0,1] \to \mathbb{S}$ sending $t \mapsto e^{2\pi i/t}$ would still be analytic but at 0 it has an essential singularities which prevents the limit to exist) and that one can always extend f uniquely to a morphism $K \to X$, i.e., the limit $\lim_{x\to 0} f(x)$ always exist when X is projective (or more generally, complete). Let us prove this in the simplest case:

Proposition 3.17. *Let $f: K \setminus 0 \to \mathbb{P}^1(K)$ be an morphism of algebraic varieties. Then f extends uniquely to a morphism $\tilde{f}: K \to \mathbb{P}^1(K)$.*

Proof. Intuitively, we can write $f(u) = [g(u): h(u)]$ for some polynomials $g, t \in K[t]$ and every $u \in K \setminus 0$. Now if either $g(0) \neq 0$ or $f(0) \neq 0$ the point $f(0)$ is well-defined. If on the other hand $h(0) = g(0) = 0$ this means that both h, g are divisible by some power of t, we let t^N be the maximal power dividing both polynomials. Then $[g(u) \dots h(u)] = [u^N g_0(u) \dots u^M h_0(u)] = [g_0(u) \dots h_0(u)]$ and we can assume without loss of generality that $g_0(0) \neq 0$. Hence this is again well-defined at 0, which shows that the morphism extends.

Let us now make this rigorous and let us unpack the statement. Let t be the coordinate of K vanishing at 0 and let U_0, U_1 be the standard open cover of $\mathbb{P}^1(K)$. Let $f^{-1}(U_i) =: V_i \subset K \setminus 0$, which are an open subsets, and pick $\tilde{V} \subset K \setminus 0$ be a small open subset such that $\tilde{V} \subset V_0 \cap V_1$. Finally, let $V \subset K$ be the open subset $\tilde{V} \cup \{0\}$.

Then by the sheaf property it is enough to show that $f_{|\tilde{V}}: \tilde{V} \to \mathbb{P}^1(K)$ extends to V. But $f(\tilde{V}) \subset U_i$ for $i = 1, 2$ and we can see both as a morphism of affine algebraic varieties. It is enough then to show that either $f_0: \tilde{V} \to U_0$ extends to $V \to U_0$ or that $f_1: \tilde{V} \to U_1$ extends to $V \to U_1$.

Since $V = \mathrm{mSpec}(K[x]_{x \cdot P(x)})$ where $P(x) = (x - a_1) \cdots (x - a_k)$ for some $a_i \neq 0$ and $U_0 = \mathrm{mSpec}(K[x_1/x_0])$ we know that f_0 is determined by an algebra morphism $K[x_1/x_0] \to K[x]_{x \cdot P(x)}$ which sends x_1/x_0 to some $g(x) = x^n g_0(x)$ with $g_0(0) \neq 0$ and $n \in \mathbb{Z}$. This simply means that the map sends $v \in V$ to $[1: v^n g_0(v)]$. Now, if $n \geq 0$ then $g(x) \in K[x]_{(x)}$ hence $g(x) \in \Gamma(V, \mathcal{O}_V)$ by Theorem 2.10, which means that it is the restriction of a unique morphism $V \to U_0$. If on the other hand $n < 0$ this means that $\tilde{V} \to U_0$ has a pole at 0. To extend it, let us look at it as a function $f_1: \tilde{V} \to U_1$. But then f_1 must correspond to the algebra morphism $K[x_0/x_1] \to K[x]_{x \cdot P(x)}$ which sends x_0/x_1 to $g(x)^{-1} = x^{-n} g_0(x)^{-1}$ with still $g_0(0) \neq 0$. But now $g(x)^{-1} \in K[x]_{(x)}$ because $-n > 0$, hence it must extend as before (we are simply rewriting the map as $v \mapsto [v^{-n} g_0(v)^{-1}: 1]$ and sending 0 to $[0: 1] = U_1 \setminus U_0$). □

Remark 3.18. It is fundamental that U has dimension one, otherwise the limit might not be unique anymore. For example, let $U = K^2 \setminus 0$ and consider the map $f : U \to \mathbb{P}^1(K)$ sending (x_0, x_1) to $[x_0 : x_1]$ which is well-defined. Then this cannot be extended to a map $K^2 \to \mathbb{P}^1(K)$. To prove this, assume that it does, and let $f(0)$ be the value of the extension at 0. But the restriction of f to any line $V(y - \alpha x) \setminus 0$ is the constant value $[1 : \alpha]$. So if the extension existed, all these values must also be equal to $f(0)$, which is absurd. There is however a natural construction to extend the map at zero: the blow-up.

4 Lecture V/VI

In this lecture we shall introduce schemes. Recall that if R is any commutative ring with unity we let $\mathrm{Spec}(R)$ be the set of its prime ideals, endowed with the Zariski topology. More precisely, all closed subsets are of the form

$$V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) : I \subset \mathfrak{p}\}$$

for any ideal $I \subset R$. Recall also that if $f \in R$ then $D(f) = \mathrm{Spec}(R) \setminus V(f)$ is called a principal open subset; these subsets form a basis for the Zariski topology on $\mathrm{Spec}(R)$, and the natural map $R \to R_f$ induces a homeomorphism $\mathrm{Spec}(R_f) \cong D(f)$.

Our main result of this lecture is the following:

Theorem 4.1. *Given any R-module M there is a unique sheaf $\tilde{M}$ on* $\mathrm{Spec}(R)$ *such that*

$$\Gamma(D(f), \tilde{M}) \cong M_f = M \otimes_R R_f \qquad \text{for every } f \in R.$$

We follow the proof in the Stacks project as we find it the most conceptual. Note that if R is integral and $M = R$, then basically the same proof of Theorem 2.10 would work. We shall prove Theorem 4.1 in various steps.

Step 1: some basics on fundamental opens Consider the set $\mathscr{B} = \{D(f)\}_{f \in R}$. This forms a basis for the topology, and by quasi-compactness of $\mathrm{Spec}(R)$ we also know that every open subset is a finite union of principal open subsets and that $\mathscr{B}$ is closed under finite intersections.

Lemma 4.2. *We have $D(g) \subset D(f)$ if and only if f is invertible in R_g, if and only if $g^e \in (f)$ for some $e \geq 1$. In this case, for every R-module M there is a natural induced map $M_f \to M_g$, which is an isomorphism whenever $D(g) = D(f)$.*

Proof. In fact, $D(g) = \{\mathfrak{p} \subset R : g \notin \mathfrak{p}\} = \mathrm{Spec}(R_g)$. If $f \in R_g$ is not invertible there must be a maximal ideal $\mathfrak{m} \subset R_g$ such that $f \in \mathfrak{m}$, i.e., $\mathfrak{m} \in D(g)$ but $\mathfrak{m} \notin D(f)$, a contradiction. If f is invertible in R_g, we can write $1 = (f/1) \cdot (h/g^n)$, i.e. $g^k(g^n - fh) = 0$ in R for some k, which shows the claim. Finally, if $g^e = fh$ for some $h \in R$ and $g \notin \mathfrak{p}$ then clearly $f \notin \mathfrak{p}$ either, and so $D(g) \subset D(f)$.

For the other statements: since f is invertible in R_g, the universal property of localization yields a unique map $R_f \to R_g$. The map $M_f \to M_g$ follows either again from the universal property or by noting that $M_f = M \otimes_R R_f$ and using the previous map. □

Step 2: reinterpretation of sheaf conditions Let $\mathscr{F}$ be a presheaf of abelian groups on a topological space X. Let $U \subset X$ be open, and let $\{U_i\}_{i\in I}$ be an open cover of U; put $U_{ij} = U_i \cap U_j$.

Lemma 4.3. *The presheaf $\mathscr{F}$ is a sheaf if and only if for every such U and cover $\{U_i\}$ the sequence*

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathscr{F}(U_i) \xrightarrow{\beta} \prod_{i,j\in I} \mathscr{F}(U_{ij})$$

is exact, where $\alpha = \prod \rho_{UU_i}$ and

$$\beta(s)_{ij} = (s_i)_{|U_{ij}} - (s_j)_{|U_{ij}}.$$

Proof. Injectivity at the first term is axiom (II) for sheaves; exactness in the middle is axiom (I). □

Step 3: sheaves on a basis of opens Let X be a quasi-compact topological space and let $\mathscr{B}$ be a basis for the topology closed under finite intersections.

Definition 4.4. A sheaf of abelian groups $\tilde{\mathscr{F}}$ on $\mathscr{B}$ is the data of an abelian group $\tilde{\mathscr{F}}(U)$ for each $U \in \mathscr{B}$ such that for every $U \in \mathscr{B}$ and every *finite* open cover $U = \bigcup_{i=1}^n U_i$ with $U_i \in \mathscr{B}$, the sequence of Lemma 4.3 is exact.

Proposition 4.5. *In the situation above, every sheaf $\tilde{\mathscr{F}}$ on $\mathscr{B}$ extends uniquely to a sheaf $\mathscr{F}$ on X satisfying $\tilde{\mathscr{F}}(U) = \mathscr{F}(U)$ for all $U \in \mathscr{B}$.*

Proof. For any $x \in X$ define

$$\mathscr{F}_x = \varinjlim_{x\in U\in\mathscr{B}} \tilde{\mathscr{F}}(U),$$

and define the sheaf $\mathscr{F}$ by the usual sheafification-by-germs recipe: for $V \subset X$ open,

$$\mathscr{F}(V) := \left\{ (s_x)_{x\in V} \in \prod_{x\in V} \mathscr{F}_x \;\middle|\; \begin{array}{l} \text{for every } x \in V \text{ there exist } U \in \mathscr{B},\ x \in U \subset V, \\ \text{and } S \in \tilde{\mathscr{F}}(U) \text{ such that } s_y = S_y \text{ in } \mathscr{F}_y \text{ for all } y \in U \end{array} \right\}.$$

This is a sheaf, and by construction it restricts to $\tilde{\mathscr{F}}$ on $\mathscr{B}$. □

Moreover, note that the stalks of $\mathscr{F}$ and $\tilde{\mathscr{F}}$ agree.

Remark 4.6. Given two sheaves $\mathscr{F}, \mathscr{G}$ on X, to describe a morphism $f : \mathscr{F} \to \mathscr{G}$ it suffices to define it on the basis $\mathscr{B}$ compatibly.

Final step All in all, we only need to prove that for any R-module M the assignment $D(f) \mapsto M_f$ defines a sheaf on the basis of principal opens. So pick a finite covering $D(f) = \bigcup_{i=1}^{n} D(g_i)$; we need to prove that

$$0 \longrightarrow M_f \longrightarrow \bigoplus_i M_{g_i} \longrightarrow \bigoplus_{i,j} M_{g_i g_j}$$

is exact. Note that $g_1, \dots, g_n$ generate the unit ideal of R_f. It is enough to prove:

Proposition 4.7. *Let R be a ring, M an R-module, and $g_1, \dots, g_n \in R$ generate the unit ideal. Then*

$$0 \longrightarrow M \longrightarrow \bigoplus_i M_{g_i} \longrightarrow \bigoplus_{i,j} M_{g_i g_j}$$

is exact.

Proof. It suffices to check exactness after localizing at each maximal ideal $\mathfrak{m}$. Since the g_i generate the unit ideal, we can assume $g_1 \notin \mathfrak{m}$, i.e. g_1 is a unit in $R_\mathfrak{m}$. Then $(M_{g_1})_\mathfrak{m} = M_\mathfrak{m}$ and $(M_{g_1 g_i})_\mathfrak{m} = (M_\mathfrak{m})_{g_i}$. Thus we may assume $g_1 = 1$. The first map is then injective. For exactness in the middle, if $(m_i)_i \in \bigoplus_i M_{g_i}$ maps to zero, note $m_1 \in M$, and $m_j - m_1 = 0$ in M_{g_j} for all j, which proves the claim. ☐

This completes the proof of Theorem 4.1. If we pick $M = R$ we get a sheaf of rings on $\operatorname{Spec}(R)$, usually denoted by $\mathscr{O}_{\operatorname{Spec}(R)}$. This should be considered the sheaf of regular functions on $\operatorname{Spec}(R)$.

Proposition 4.8 (Stalks). *For any $x \in \operatorname{Spec}(R)$ and any sheaf $\tilde{M}$ on $\operatorname{Spec}(R)$ coming from an R-module M, we have*

$$\tilde{M}_x \cong M_\mathfrak{p}$$

where $\mathfrak{p}$ is the prime ideal corresponding to x. In particular, the stalk of the structure sheaf $\mathscr{O}_{\operatorname{Spec}(R)}$ is the local ring $R_\mathfrak{p}$.

Proof. Note that $\mathscr{B}_x := \{D(f) : x \in D(f)\}$ is a cofinal system of open neighborhoods. Hence

$$\tilde{M}_x = \varinjlim_{D(f) \in \mathscr{B}_x} \tilde{M}(D(f)) = \varinjlim_{D(f) \in \mathscr{B}_x} M_f.$$

Now $x \in D(f)$ iff $f \notin \mathfrak{p}$. Ordering $\{f \in R : f \notin \mathfrak{p}\}$ by $f \leq g$ if $D(g) \subset D(f)$, we have transition maps $M_f \to M_g$, and

$$\varinjlim_{D(f)\in\mathscr{B}_x} M_f = \varinjlim_{f\notin\mathfrak{p}} M_f.$$

There are natural maps $M_f \to M_\mathfrak{p}$ for $f \notin \mathfrak{p}$, inducing $\varinjlim_{f\notin\mathfrak{p}} M_f \to M_\mathfrak{p}$. This is injective: if $m/1 \in M_f$ maps to 0 in $M_\mathfrak{p}$, then some $g \notin \mathfrak{p}$ satisfies $gm = 0$, so m maps to 0 in M_{gf}. Surjectivity is clear. □

Corollary 4.9. *For any ring R, the space* $\operatorname{Spec}(R)$ *with* $\mathcal{O}_{\operatorname{Spec}(R)}$ *is a locally ringed space.*

Definition 4.10. An *affine scheme* is a locally ringed space isomorphic to

$$(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$$

for some ring R. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme, i.e. there is an open cover $X = \bigcup_i U_i$ such that each $(U_i, \mathcal{O}_{X|U_i})$ is affine.

Remark 4.11. If $(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ is an affine scheme and $f \in R$, then we can still consider f as a function on $\operatorname{Spec}(R)$, with the caveat that the codomain varies: the value at $\mathfrak{p} \in \operatorname{Spec}(R)$ is the image of f in $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = \kappa(x)$, the residue field of the corresponding point x.

For example, if R is a finitely generated integral K-algebra with K algebraically closed, then for a maximal ideal $\mathfrak{m}$ corresponding to $x \in \operatorname{mSpec}(R)$ we recover $f(x)$ via $f \equiv f(x) \bmod \mathfrak{m}$. For K-algebras, such an f yields a function $\operatorname{mSpec}(R) \to K$. On the other hand, the value at the generic point $(0) \in \operatorname{Spec}(R)$ is f viewed in $\operatorname{Frac}(R)$.

Similarly, if $Z \subset \operatorname{mSpec}(R)$ is irreducible with prime $\mathfrak{p}$, then $f_{|Z}$ is a regular function on Z, i.e. an element of $R/\mathfrak{p}$, and the "value'' at $\mathfrak{p}$ is that function seen in $\operatorname{Frac}(R/\mathfrak{p})$ (the function field of Z). In particular, $f(\mathfrak{p}) = 0$ iff f vanishes on Z.

4.1 Morphisms of schemes

A morphism of schemes is a morphism of locally ringed spaces.

Proposition 4.12 (Morphisms to affine schemes). *Let $(X, \mathcal{O}_X)$ be a locally ringed space and $Y = \operatorname{Spec}(R)$ an affine scheme. Then there is a natural bijection*

$$\operatorname{Hom}_{l.r.s}(X, Y) \cong \operatorname{Hom}_{\text{rings}}(R, \Gamma(X, \mathcal{O}_X)).$$

Let $f \in \mathrm{Hom}(X, Y)$; then $f : X \to Y$ is continuous and $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$. The map $f^\#$ induces $\psi_f : R \to \Gamma(X, \mathcal{O}_X)$, and $f \mapsto \psi_f$ is the bijection above.

Lemma 4.13. *For any $x \in X$ consider the composition*

$$R \xrightarrow{\psi_f} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$$

and let $\mathfrak{p} \subset R$ be the inverse image of $\mathfrak{m}_x$. Then $f(x) = \mathfrak{p}$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \Gamma(X, \mathcal{O}_X) & \longrightarrow & \mathcal{O}_{X,x} \\ {\scriptstyle \psi_f}\uparrow & & \uparrow \\ R & \longrightarrow & R_{\mathfrak{p}'} \end{array}$$

where $\mathfrak{p}' = f(x)$. The right vertical map is local, so the preimage of $\mathfrak{m}_x$ is $\mathfrak{p}'$. Commutativity gives $\mathfrak{p}' = \mathfrak{p}$. □

Lemma 4.14 (Generalization of principal open subsets). *Let $(X, \mathcal{O}_X)$ be a locally ringed space and $f \in \Gamma(X, \mathcal{O}_X)$. Then $D(f) = \{x \in X : f_x \notin \mathfrak{m}_x\}$ is open, and $f_{|D(f)}$ is invertible.*

Proof. Fix $x \in D(f)$. Since $f_x \notin \mathfrak{m}_x$, choose $g_x \in \mathcal{O}_{X,x}$ with $f_x g_x = 1$. For $x \in U$ small, represent g_x by $g \in \mathcal{O}_X(U)$. Then $gf - 1 \in \mathcal{O}_X(U)$ vanishes at x, hence vanishes on some open $V \subset U$; thus $V \subset D(f)$, and $f_{|V}$ is invertible. □

Proof of Proposition 4.12. We construct the inverse to $f \mapsto \psi_f$. Given $\psi \in \mathrm{Hom}(R, \Gamma(X, \mathcal{O}_X))$, define $f : X \to \mathrm{Spec}(R)$ by $x \mapsto \psi^{-1}(\mathfrak{m}_x)$. This is continuous since for $D(g) \subset \mathrm{Spec}(R)$ we have

$$f^{-1}(D(g)) = \{x : \psi(g) \notin \mathfrak{m}_x\} = D(\psi(g)).$$

To get $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$, it is enough to define on principal opens: for $D(g) \subset Y$ set

$$\mathcal{O}_Y(D(g)) = R_g \longrightarrow \mathcal{O}_X(D(\psi(g)))$$

using that $\psi(g)$ is invertible on $D(\psi(g))$. The induced stalk maps $R_{\mathfrak{p}} \to \mathcal{O}_{X,x}$ are local because $\mathfrak{p} = \psi^{-1}(\mathfrak{m}_x)$. □

Corollary 4.15. *The category of affine schemes is equivalent to the opposite category of rings.*

Relative point of view One core philosophy is relativity: instead of studying schemes over a fixed field, we study schemes over a base scheme.

Definition 4.16. Let S be a scheme. An *S-scheme* is a scheme X together with a morphism $X \to S$. A morphism of S-schemes is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ S & \xrightarrow{\mathrm{Id}_S} & S \end{array}$$

Since $\mathbb{Z}$ is initial in rings, every scheme is naturally a scheme over $\mathrm{Spec}(\mathbb{Z})$. If R is a K-algebra, then $\mathrm{Spec}(R)$ is a $\mathrm{Spec}(K)$-scheme. Many properties are relative (properties of the morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(K)$), others are absolute.

In general, think of an S-scheme as a family of schemes parametrized by S. This will be clear once we introduce fibre products.

4.2 Sheaves of modules

If $(X, \mathcal{O}_X)$ is a ringed space, a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is a sheaf on X with $\mathcal{F}(U)$ an $\mathcal{O}_X(U)$-module for each open U, compatibly with restriction. For every ring R and R-module M we have the associated sheaf $\tilde{M}$ on $\mathrm{Spec}(R)$.

Proposition 4.17. *The association $M \mapsto \tilde{M}$ is exact.*

Proof. Exactness can be checked on stalks and follows from Proposition 4.8. □

Every module M has a presentation $R^J \to R^I \to M \to 0$, which yields an exact sequence of sheaves

$$\mathcal{O}^J_{\mathrm{Spec}(R)} \longrightarrow \mathcal{O}^I_{\mathrm{Spec}(R)} \longrightarrow \tilde{M} \longrightarrow 0.$$

Definition 4.18 (Quasi-coherent sheaves). Let $(X, \mathcal{O}_X)$ be a locally ringed space. A quasi-coherent sheaf on X is a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ such that for every $x \in X$ there exists an open $x \in U$ and a presentation

$$\mathcal{O}^J_{X|U} \longrightarrow \mathcal{O}^I_{X|U} \longrightarrow \mathcal{F}_{|U} \longrightarrow 0.$$

Thus, quasi-coherent sheaves are locally built from $\mathcal{O}_X$ using generators and relations.

Theorem 4.19. *Let X be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. The following are equivalent:*

1. *$\mathscr{F}$ is quasi-coherent.*

2. *For every open affine $U = \operatorname{Spec}(R)$ and $f \in R$, the natural map*

$$\Gamma(U, \mathscr{F})_f \longrightarrow \Gamma(D(f), \mathscr{F})$$

   is an isomorphism.

3. *For every open affine $U = \operatorname{Spec}(R)$ in X there exists an R-module M with $\mathscr{F}_{|U} \cong \widetilde{M}$.*

Before the proof: in (2), since $\mathscr{F}$ is an $\mathscr{O}_X$-module, $\Gamma(D(f), \mathscr{F})$ is an R_f-module; by the universal property of localization we get the map $\Gamma(U, \mathscr{F})_f \to \Gamma(D(f), \mathscr{F})$.

Proof. (1)⇒(2): we may assume X affine. The statement is clear for $\mathscr{F} = \tilde{M}$. Cover X by finitely many $D(g_i)$ so that each $\mathscr{F}_i = \mathscr{F}|_{D(g_i)}$ has a presentation; then $\mathscr{F}_i = \widetilde{M_i}$ for suitable M_i and satisfies (2); similarly for pairwise intersections. Using the exactness sequences for sheaves and exactness of localization, a diagram chase shows the desired isomorphism.

(2)⇒(3): assume $X = U$ is affine and put $M = \Gamma(X, \mathscr{F})$. Then $\Gamma(X, \mathscr{F})_f = M_f = \Gamma(D(f), \mathscr{F})$ for all f, hence $\mathscr{F} = \tilde{M}$ by the defining property of $\tilde{M}$.

(3)⇒(1) is obvious from the definition. □

Corollary 4.20. *If $X = \operatorname{Spec}(R)$ is affine, the functor $M \mapsto \tilde{M}$ induces an equivalence between R-modules and quasi-coherent sheaves on X.*

Let now $\phi : R \to S$ be a map of rings and let M be an S-module. Viewing M as an R-module via ϕ, denote it by M'.

Proposition 4.21. *Let $f : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ be the induced map. Then $f_*(\tilde{M}) \cong \widetilde{M'}$.*

Proof. For $g \in R$ we have

$$\Gamma(D(g), f_*\tilde{M}) = \Gamma(D(\phi(g)), \tilde{M}) = M_{\phi(g)} = \Gamma(D(g), \widetilde{M'}).$$

□

4.3 Open and closed immersions

The two most basic morphisms of schemes (or locally ringed spaces in general) are open and closed immersions.

Definition 4.22. A map of locally ringed spaces $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is

1. an open immersion if $f : X \to Y$ is a homeomorphism onto an open subset and $f^{-1}\mathcal{O}_Y = \mathcal{O}_X$ (intuitively, functions on opens of X are just functions on the same open of Y);

2. a closed immersion if $f : X \to Y$ is a homeomorphism onto a closed subset and $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective (intuitively, functions on X are locally restrictions of functions on Y).

For example, if $R \to R_f$ is a localization, then $\mathrm{Spec}(R_f) \to \mathrm{Spec}(R)$ is an open immersion onto $D(f)$. More generally, for a multiplicative set $S \subset R$, the induced map $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ identifies $\mathrm{Spec}(S^{-1}R)$ with $\bigcap_{s\in S} D(s)$. If this intersection is infinite, it need not be open; in particular, not every localization map yields an open immersion onto its image unless the intersection is open.

Closed immersions are more interesting, and nilpotents naturally appear:

Proposition 4.23. *Let R be a ring and $I \subset R$ an ideal. Then* $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ *is a closed immersion. Moreover, every closed immersion of affine schemes is of this form.*

Proof. Topologically, $\mathrm{Spec}(R/I)$ identifies with $V(I) \subset \mathrm{Spec}(R)$. On stalks, the induced map $R_\mathfrak{p} \to (R/I)_\mathfrak{p}$ is surjective for $\mathfrak{p} \supset I$, and is zero for $\mathfrak{p} \not\supset I$, hence $\mathcal{O}_R \to f_*\mathcal{O}_{\mathrm{Spec}(R/I)}$ is surjective. Conversely, if $\iota : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a closed immersion, then $\mathcal{O}_{\mathrm{Spec}(R)} \to \iota_*\mathcal{O}_{\mathrm{Spec}(S)}$ is surjective; but $\iota_*\mathcal{O}_{\mathrm{Spec}(S)} = \tilde{S}$ where we see S as an R-module, and the associated map of rings $\phi : R \to S$ is surjective since all induced maps $R_\mathfrak{p} \to S_\mathfrak{p}$ are surjective (Atiyah–Macdonald, 3.9). $\square$

Example 4.24. Let R be a ring, $\mathfrak{p} \subset R$ prime, $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(R/\mathfrak{p})$. The surjection $R \to R/\mathfrak{p}$ induces a closed immersion $Y = V(\mathfrak{p}) \subset X$. Also $R \to R/\mathfrak{p}^2$ induces a closed immersion $Y' = \mathrm{Spec}(R/\mathfrak{p}^2) \to X$, factoring $Y \to Y' \to X$. The kernel of $R/\mathfrak{p}^2 \to R/\mathfrak{p}$ is $\mathfrak{p}/\mathfrak{p}^2$, a natural quasi-coherent $\mathcal{O}_{Y'}$-module. Imagine that X is a $\mathcal{C}^\infty$-manifold and that Y is a submanifold. Let $\mathcal{I}_Y \subset \mathcal{C}^\infty_X$ be the ideal sheaf of functions vanishing along Y. Show that $\mathcal{I}_Y/\mathcal{I}_Y^2$ is naturally isomorphic to the normal bundle of Y in X. In this sense, nilpotens see infinitesimal information on how Y is embedded into X.

4.4 Proj and graded rings

An extremely useful way to construct non-affine schemes is the *Proj*-construction. Let $R = \bigoplus_{d\geq 0} R_d$ be a graded ring and $R_+ = \bigoplus_{d\geq 1} R_d$ the irrelevant ideal. An ideal $I \subset R$ is *homogeneous* if $I = \bigoplus_{d \geq 0}(I \cap R_d)$ (equivalently, if it is generated by homogeneous elements). Define

$$\mathrm{Proj}(R) = \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ homogeneous prime}, R_+ \not\subseteq \mathfrak{p}\}.$$

For homogeneous $f \in R$ of degree > 0, put

$$D_+(f) = \{\mathfrak{p} \in \mathrm{Proj}(R) : f \notin \mathfrak{p}\}.$$

These $D_+(f)$ are principal opens and form a basis; they cover $\mathrm{Proj}(R)$.

For $\mathfrak{p} \in \mathrm{Proj}(R)$, let $R_\mathfrak{p}$ be the localization of R at the multiplicative system of homogeneous elements not in $\mathfrak{p}$. This inherits a $\mathbb{Z}$-grading, and we let $R_{(\mathfrak{p})}$ be its degree-zero subring. For homogeneous f of positive degree, $R_{(f)}$ denotes the degree-zero subring of R_f.

We define a sheaf of rings $\mathcal{O}_{\mathrm{Proj}(R)}$ by declaring

$$\mathcal{O}_{\mathrm{Proj}(R)}(D_+(f)) := R_{(f)} \qquad \text{for homogeneous } f \text{ of deg } > 0,$$

with the obvious restriction maps.

Theorem 4.25 (Basic properties of Proj). *With the notation above:*

1. *The opens* $D_+(f)$ *cover* $\mathrm{Proj}(R)$.
2. *For every* $\mathfrak{p} \in \mathrm{Proj}(R)$*, the stalk* $\mathcal{O}_{\mathrm{Proj}(R),\mathfrak{p}}$ *is canonically isomorphic to the local ring* $R_{(\mathfrak{p})}$.
3. *There is a natural isomorphism of schemes*

$$\left(D_+(f), \mathcal{O}_{\mathrm{Proj}(R)}|_{D_+(f)}\right) \cong \left(\mathrm{Spec}(R_{(f)}), \mathcal{O}_{\mathrm{Spec}(R_{(f)})}\right).$$

In particular, $\mathrm{Proj}(R)$ *is a scheme.*